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On construction schemes: Building the uncountable from finite pieces

T E S I S

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Jorge Antonio Cruz Chapital
jorgeacruzchapital@hotmail.com

Asesores:
Dr. Osvaldo Guzmán González
Centro de Ciencias Matemáticas, UNAM.

Dr. Michael Hrušák
Centro de Ciencias Matemáticas, UNAM.

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Abstract

On construction schemes: Building the uncountable from finite pieces

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Resumen

En esta tesis se desarrollará un análisis estructural de los esquemas de construción introducidos en [112]. La importancia de este estudio se verá reflejada al construir una gran cantidad de objetos combinatorios distintos que han sido de gran interés en las matemáticas. Tambien se continuará con el estudio de los axiomas de captura asociados a esquemas de construcción. De dichos axiomas se construirán varios objetos cuya existencia se sabe independiente de los axiomas usuales de la teoría de conjuntos.

Palabras Clave: uncountable, construction scheme, diamond principle, morasses, capturing, trees, gap, coloring.

Abstract

In this thesis, a structural analysis of construction schemes (as introduced in [112]) is developed. The importance of this study will be justified by constructing several distinct combinatorial objects which have been of great interest in mathematics. We then continue the study of capturing axioms associated to construction schemes. From them, we construct several uncountable structures whose existence is known to be independent from the usual axioms of Set Theory.

Keywords: uncountable, construction scheme, diamond principle, morasses, capturing, trees, gap, coloring.

Contents

Introduction

Throughout history, infinity has captivated and amazed humanity, sparking curiosity and contemplation among generations. A fabulous trip lasting more than 2000 years lead, in the 19th century, to beautiful results regarding this concept. Georg Cantor delved deeper into the nature of infinity by proving, among other things, that infinite sets may have different "sizes". Formally, he showed that there can not be a bijective function from the natural to the real numbers. Since then, a great part of effort has been put into understanding the mysteries lying in the realm surrounding the two aforementioned infinities. In this thesis, we will explore ω_1, the first uncountable cardinal lying in *between ω and the hill of treasures* that are the real numbers.

The purpose of this thesis is to study ω_1 by further developing the theory of construction schemes as introduced by Stevo Todorčević in [112]. By means of descriptive set-theoretic results, it is often impossible to deduce the existence of interesting objects of size ω_1 by presenting them throughout "nice and simple" definitions. For that reason, there has been a lot of interest in presenting and studying methods for constructing such objects:

- Probably the most straightforward approach is to build them by transfinite recursion in ω_1 many steps. Typical examples of this method are the construction of an Aronszajn tree from [54] or [64] and the construction of a Hausdorff gap from [91] or [125].

- A second approach would be to show the consistency of an uncountable object with the desired properties, analyze the complexity of the sentences defining such properties, and then appeal to Jerome H. Keisler's completeness theorem for $L_\omega^\omega(\mathcal{Q})$ (see [61]) to conclude that such an object in fact exists. An example of this method is the coherent family of functions from [36]. A variation of this method involving the $\Diamond$-principle was presented by Menachem Magidor and Jerome Malitz in [72].

- A third approach is by using the method of walks on ordinals and ordinal metrics as introduced by Tordorčević (see [110]). Historically, this method has been proved to be one of the most useful when dealing with constructions of uncountable objects. Examples of this are the solution of the famous L-space problem from [76] and the construction of a "rainbow coloring" from [120].

- A fourth approach, which is arguably the most related to the present work, is to

construct the desirable uncountable object by finite approximations through the use of simplified morasses as introduced by Ronald Jensen (see [26]) and later simplified by Daniel Velleman in [128].

Just as simplified morasses, construction schemes are objects that let us build uncountable objects through finite approximations. Roughly speaking, a construction scheme $\mathcal{F}$ is a collection of finite subsets of ω_1 with strong coherent properties (see Definition 2.2.2). These coherent properties allow us to build directed families of finite structures indexed by the elements of the scheme. Such structures are built recursively by amalgamation processes dictated by the construction scheme. In the end, the uncountable object that we construct is in some sense the direct limit of the family that we defined. The main difference between construction schemes and morasses is that in morasses we only amalgamate two structures at a given time, whereas in construction schemes, the number of amalgamations may vary.

That same attribute which sets appart construction schemes and morasses, allow construction schemes to *consistently* have further properties, powerful enough to imply the existence of a large amount of objects which are known to exist under extra set-theoretic assumptions. These properties, which we call *n-capturing*, *capturing* and *fully capturing*, can be described as "finitizations" (relative to a construction scheme) of the well-known $\diamondsuit$-prinicple (see 2.4.8). The capturing axioms CA_n, CA and FCA assert the existence of construction schemes which are n-capturing, capturing and fully capturing respectively. All of these axioms are implied by the $\diamondsuit$-principle and also hold in any forcing extension of the universe obtained by adding at least ω_1 many Cohen reals. Given an n-capturing construction scheme $\mathcal{F}$ we also can define the n-parameterized Martin's number associated to it as $\mathfrak{m}_{\mathcal{F}}^n$. This cardinal invariant is defined as the Martin's number corresponding to the family of *ccc*-forcing notions which force our construction scheme to be n-capturing. Even though this cardinal is strictly related to a combinatorial structure over ω_1 (the construction scheme), the assertion "$\mathfrak{m}_{\mathcal{F}}^n > \omega_1$" is strong enough to imply the existence of objects such as Ramsey ultrafilters over ω. Furthermore, as we will show, there are meaningful statements about uncountable objects which Martin's axiom can not decide, that are true under "$\mathfrak{m}_{\mathcal{F}}^n > \omega_1$", and which are false under axioms such as PFA, and CH (so in particular, $\diamondsuit$-principle).

Structure of the thesis

In this thesis we will develop to a great extent the theory of construction schemes and their structural properties, and we will obtain important results about set-theoretic assumptions associated with them. The importance of this study will be justified by constructing several distinct combinatorial objects which have been of a great interest to set theorists and topologists. A large amount of these objects will be constructed in a rather different form from the original constructions, and some of them will make their first appearance here.

In **Chapter 1** we will present some of the preliminaries, and fix the notation that will be used through out the thesis.

Chapter 2 is intended to serve as an introduction to the theory of ordinal metrics and construction schemes over ω_1. In this chapter, we will define these two notions and analyze their general behavior. In particular, we will prove that construction schemes are in a one-to-one correspondence between certain kinds of ordinal metrics. We will also define the notions of n-capturing, capturing, and fully capturing construction schemes, as well as the capturing axioms CA_n, CA, and FCA associated to them. Finally, we will introduce the parametrized Martin's axioms $\mathfrak{m}_{\mathcal{F}}^n$ associated to n-capturing construction schemes. Along the way, we will present the main theorems regarding the existence of distinct types of construction schemes. Nevertheless, such theorems will be proved and discussed in further detail until Chapters 4 and 5.

In **Chapter 3** we will provide a handful of applications of construction schemes both in ZFC and under extra set-theoretic assumptions. These applications are of interest in set theory, topology, infinite combinatorics, algebra, and analysis. The constructions are mostly independent of each other, so that the reader can start from the ones that she or he finds more interesting. For the convenience of the reader, we list here some of the constructions that appear in this chapter. References for further study and historic remarks will be provided as we encounter them.

1. **Hausdorff gaps** An interesting feature of the Boolean algebra $\mathscr{P}(\omega)/\text{FIN}$ is that it is not complete. The easiest way to see this is as follows: Let $\{A_\alpha : \alpha \in 2^\omega\} \subseteq [\omega]^\omega$ be an AD family. For every $H \subseteq 2^\omega$, consider the set $\mathcal{A}_H = \{[A_\alpha] : \alpha \in H\}$ (where $[A_\alpha]$ is the class of A_α in $\mathscr{P}(\omega)/\text{FIN}$). It follows form a simple counting argument that there must be an $H \subseteq 2^\omega$ such that $\mathcal{A}_H$ has no supremum. Although this is a very simple argument, we are often interested in more concrete examples of the incompleteness of $\mathscr{P}(\omega)/\text{FIN}$. The nicest examples are provided by gaps. Moreover, gaps are important because they represent obstructions that we may encounter when embedding structures in $\mathscr{P}(\omega)/\text{FIN}$ (a very illustrative example of this situation is Theorem 8.8 of [121]). The classic construction of a Hausdorff gap requires a very clever argument to take care of 2^ω many tasks in only ω_1 many steps. With construction schemes, we will be able to provide a very simple construction of a Hausdorff gap, which under extra set-theoretic assumptions satisfies some interesting and (until now) unstudied combinatorial properties.

2. **Luzin-Jones almost disjoint families.** This topic is about almost disjoint families (AD) and their separation properties. A *Luzin family* is an AD family of size ω_1 in which no two uncountable subfamilies can be separated. On the other hand, a Jones family is an AD family with the property that every countable subfamily of it can be separated from its complement. It is easy to prove that both of this kind of families exist. However, building a family that is both Luzin and Jones at the same time is much more complicated, since there is a tension between this two properties. A very difficult and highly complex construction

of a Luzin-Jones family appears in [42]. With construction schemes, we will be able to build an almost disjoint family satisfying strong properties related to a concept that we will call *the Luzin representation of a partial order*. From these properties we will not only be able to show that the constructed family is in fact Luzin-Jones, but from it, we will derive the existence of a handful of objects related to gaps. In particular, we will generalize, in strong way, the results of Hausdorff regarding the existence of gaps in $\mathscr{P}(\omega)/\mathrm{FIN}$. It is worth pointing out that Luzin-Jones families can also be used to build interesting examples in functional analysis (see [39]).

3. **Donut-inseparable gaps.** Naively speaking, we say that gap over $\mathscr{P}(\omega)\backslash\mathrm{FIN}$ is donut-inseparable if it can be reconstructed from two disjoint inseparable subfamilies of an *AD* family. We will show, without assuming any extra axioms, that there are some gaps which are donut-inseparable and some which are not. This result suggest the following weakining of the previously defined notion: A gap is strongly donut-separable if each cofinal subgap is donut-separable. With the help of construction schemes and their parametrized Martin's axioms, we will show that the statement "There are no strongly donut-separable gaps" is independent from ZFC.

4. **Gap cohomology groups.** In [105], Talayco defined what it means for two (ω_1, ω_1)-gaps to be cohomologically different. This notion allowed him to define *the gap cohomology group* of an ω_1-tower. We will generalize the notion of the gap cohomology group to a greater class of substructures of $\mathscr{P}(\omega)/\mathrm{FIN}$ and show that all these groups can be as big as possible.

5. **Luzin coherent family of functions.** We now look at a generalization of the Hausdorff gaps discussed previously. A Luzin coherent family of functions is a coherent system of functions supported by a pretower, in which we impose a strong non triviality condition. The importance of these families is that they provide many cohomologically different gaps. They were first studied by Talayco in [105]. Later Farah proved that such families exist (see [36]). The proof of Farah is highly non-constructive and indirect, since it appeals to Keisler's completeness Theorem. We will build such families using a construction scheme. No previous direct construction was known.

6. **Independent coherent family of functions**. We now return to the study of gaps that are obtained from a coherent family of functions. However, this time we want our family of gaps to be "independent". This means that we can either fill or freeze any subfamily without filling or freezing any of the remaining gaps in the family. A similar result was obtained by Yorioka assuming the $\Diamond$-principle in [133] (the analogue result for Suslin trees was proved by Abraham and Shelah in [3]).

7. **Countryman lines**. Let $(X, <)$ be a total linear order. Except for the trivial cases X^2 is not a linear order. In this way, it makes sense to ask how many chains we need to cover it. A Countryman line is an uncountable linear order

whose square can be covered with only countably many chains. These orders seem so paradoxical at first glance that Countryman conjectured they do not exist. However, it was first proved by Shelah that Countryman lines do exist (see [93]).

8. **Aronszajn trees.** Aronszajn trees are the most well-known examples of the incompactness of ω_1. An Aronszajn tree is a tree of height ω_1, its levels are countable, yet it has no cofinal branches. A simple way in which we can guarantee that a tree (of height ω_1) has no cofinal branches is to make it "special", which means that it can be covered with countably many antichains (equivalently, they can be embedded in the rational numbers). Aronszajn proved this kind of trees exists.

9. **Suslin trees.** A Suslin tree is an Aronszajn tree in which every antichain is countable. The Suslin Hypothesis (SH) is the statement that there are no Suslin trees. We now know that SH is independent from ZFC. A related concept, the Suslin lines, were introduced by Suslin while studying the ordering of the real numbers. Kurepa was the one to realize that there is a Suslin line if and only if there is a Suslin tree. Applications and constructions from Suslin trees are abundant in the literature. We will use capturing schemes to build two types of these trees: Coherent Suslin and full Suslin trees. These two families of trees are diametrically opposed. While forcing with a Coherent tree, it completely destroys the *ccc*-ness of it, while with a full Suslin tree, many subtrees of it remain *ccc*.

10. **Suslin lower semi-lattices**. If in the definition of a Suslin tree, we relax the condition of being a tree to just a being a lower semi-lattice, we get the notion of a Suslin lower semi-lattice. They were introduced by Dilworth, Odell and Sari (see [28]) in the context of Banach spaces. They were then studied by Raghavan and Yorioka (see [86]). Among other things they proved that the $\lozenge$-principle implies that $\mathscr{P}(\omega)$ contains a Suslin lower semi-lattice. We were able to obtain the same result from a 2-capturing scheme.

11. **Entangled sets**. A well-known theorem of Cantor is that any two countable dense linear orders with no end points are isomorphic. The straightforward generalization to linear orders of size ω_1 is false. For this reason we want to restrict to suborders of the real numbers. We say that an uncountable $B \subseteq \mathbb{R}$ is ω_1-dense if $|U \cap \mathbb{B}| = \omega_1$ whenever U is an open interval whose intersection with U is nonempty. A remarkable theorem of Baumgartner is that PFA implies that any two ω_1-dense sets of reals are isomorphic (see [11]). This statement is now known as Baumgartner axiom ($BA(\omega_1)$). An entangled set is a subset of $\mathbb{R}$ with very strong combinatorial properties. The existence of an entangled set implies the failure of the Baumgartner axiom. Entangled sets were introduced by Abraham and Shelah in order to show that $BA(\omega_1)$ does not follow from Martin's axiom (see [7]). We will use a capturing scheme to build an entangled set. In this way, the existence of certain capturing schemes contradict the Baumgartner axiom.

12. *ccc* **destructible** 2-**bounded coloring without injective sets**. A coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ is called 2-bounded if every color appears at most 2 times. A set $A \subseteq \omega_1$ is c-injective if no color appears twice in $[A]^2$. Galvin was the first to wonder if there is a 2-bounded coloring without an uncountable injective set. He proved that such coloring exists assuming the Countinuum Hypothesis. On the other hand, the third author proved that no such coloring exist under PFA. years later, Abraham, Cummings and Smyth proved that MA is consistent with the existence of a 2-bounded coloring without uncountable injective sets (see [2]). After hearing this result, Friedman asked for a concrete example of a 2-bounded coloring without an uncountable injective set, but that such set can be added with a *ccc* partial order. In [2] such example is constructed assuming CH and the failure of the Suslin hypothesis. We will find an example with a 3-capturing scheme.

13. **Oscillation theory of** 2-**capturing construction schemes** In the book [121], Todorčević developed an oscillation theory which is based on an unbounded family of functions. A plethora of applications of this theory have been found throughout the years. For example, the oscillation theory is key in the proof that PFA implies that the continuum is ω_2 (see [12]). We will develop a similar theory using a 2-capturing scheme. An important difference between the classic oscillation theory and the one from capturing schemes, is that this new one is based on a bounded family of functions. Using this new oscillation theory, we can prove the existence of the following objects:

   (a) **Sixth Tukey type**. The Tukey ordering is a useful tool to compare directed partial orders. Its purpose is to study how a directed partial order behave cofinally. It was introduced by Tukey in [124] in order to study convergence in topology. The Tukey classification of countable directed partial orders is very simple: Every countable directed partial order is Tukey equivalent to 1 or to ω. The Tukey classification of directed sets of size ω_1 becomes much more interesting. We now have at leas five Tukey types: 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$. We may wonder if there is a directed partial order of size ω_1 that is not Tukey equivalent to one if this five. In [40], Isbell proved that CH entails the existence of a sixth Tukey type. This was greatly improved by Todorčević in [117], where he proved that CH implies that there are 2^{ω_1} distinct Tukey types. On the other hand, in the same paper he showed that PFA implies that there are no sixth Tukey types. Here, we found a sixth Tukey type from a capturing scheme.

   (b) $\omega_1 \nrightarrow (\omega_1, \omega + 2)_2^2$. Given $\alpha < \omega_1$, the partition $\omega_1 \rightarrow (\omega_1, \alpha)_2^2$ means that for every $c : [\omega_1]^2 \longrightarrow 2$, either there is an uncountable 0-monochromatic set, or there is a 1-monochromatic set of order type α. A celebrated result of Erdös and Rado (extending a theorem by Dushnik, Miller and Erdös) is that $\omega_1 \rightarrow (\omega_1, \omega + 1)_2^2$. We may wonder if this theorem can be improved. This turns out to be independent from ZFC. The Proper Forcing Axiom implies that $\omega_1 \rightarrow (\omega_1, \alpha)_2^2$ for every $\alpha < \omega_1$ (see [111]), while $\mathfrak{b} = \omega_1$ implies

$\omega_1 \nrightarrow (\omega_1, \omega+2)^2_2$. We will prove a similar result from our oscillation theory.

(c) **Non productivity of *ccc* partial orders**. When is the product of two *ccc* partial orders again *ccc*? This question has been of interest to set theorists for a long time. On one hand, Martin's axiom implies that the product of *ccc* partial orders is *ccc*. On the other hand, the failure of the Suslin Hypothesis implies the opposite. Consistent examples of two *ccc* partial orders whose product is not *ccc* have been constructed by Galvin under CH (see [64]) and by Todorčević under $\mathfrak{b} = \omega_1$ in [121]. As an applications of the oscillation theory that we developed, we encounter new situations in which there are *ccc* partial orders whose product is not *ccc*.

(d) **Suslin towers**. In [108], Todorčević developed an analogue of his oscillation theory, now this time based on non-meager towers. While studying this oscillation, Borodulin-Nadzieja and Chodounský introduced the notion of a Suslin tower. A tower $\mathcal{T} = \{T_\alpha : \alpha \in \omega_1\}$ is Suslin if every uncountable subset of $\mathcal{T}$ contains two pairwise $\subseteq$-incomparable elements. The existence of a Suslin tower is independent from ZFC. It can be proved that the Open Graph Axiom forbids the existence of such families, while a Suslin tower can be constructed assuming $\mathfrak{b} = \omega_1$ (see [13]). We will build a Suslin tower as an application of the oscillation theory obtained from a 2-capturing scheme.

(e) **S-spaces**. Hereditarily separable and hereditarily Lindelöf are two properties that in some sense are dual to each other. We may wonder if they always coincide. The question is only of interest in the realm of regular spaces. An S-space is a regular hereditarily separable, non Lindelöf space. The study of S-spaces (and the dual notion, L-spaces) used to be one of the most active areas of set-theoretic topology. S-spaces can be constructed under several set-theoretic hypothesis (like CH or the negation of the Suslin Hypothesis). It was Todorčević who for the first time succeeded in proving that it is consistent that there may be no S-spaces (see [121]). Here, we will apply the oscillation theory obtained from a 2-capturing scheme to construct several S-spaces.

14. **σ-monotone spaces.** Monotone and σ-monotone spaces are a particular kind of metric spaces which were defined in [83] by Aleš Nekvinda and Ondřej Zindulka. In [135], Zindulka used such spaces to prove the existence of universal measure zero sets of large Hausdorff dimension. Here, we will show that the capturing axiom CA implies the existence of a metric space of cardinality ω_1 which has no uncountable monotone subspaces.

In **Chapter 4** we will study the effect that forcing with *ccc* posets has on the capturing properties related to construction schemes. This is done for two purposes: The first one is to prove that the notions of *capturing* and *capturing with partitions* are different, and the second one is to show that the inequality $\mathfrak{m}_\mathcal{F} > \omega_1$ is consistent with ZFC. Lastly, we will prove that under this later assumption, we can deduce the existence of Ramsey ultrafilters over ω. A particularity of these results is that such ultrafilters can be explicitly defined from a combinatorial structure over ω_1 (i.e., a construction

scheme).

In **Chapter 5** we will define the forcing $\mathbb{P}(\mathcal{F})$ and use it to construct construction schemes without assuming any extra axioms. Later we will prove that Jensen's $\Diamond$-principle implies the capturing axiom $FCA(part)$. This was first claimed in [112]. However, the sketch of the proof given there is incomplete. Finally, we will show that PID implies the nonexistence of 2-capturing construction schemes.

Finally, in **Chapter 6**, we briefly discuss some open problems and future lines of research.

The following chart summarizes many of the constructions which can be carried out with a construction scheme, and that will appear in this thesis. All of the results stated in the chart are new contributions.

Object	Extra set-theoretic assumption	Result number
Donut-separable gap	-	3.1.25
Donut-inseparable gap	-	3.1.22
Almost disjoint family coding any ω_1-like order	-	3.1.13 and 3.1.20
(X, Y)-gap for all ω_1-like orders	-	3.1.21
$*$-Lower semilattices with large gap cohomology groups	-	3.1.69
Not strongly donut-separable Hausdorff gap	$\mathfrak{m}_{\mathcal{F}} > \omega_1$	3.1.36
Ramsey ultrafilter	$\mathfrak{m}_{\mathcal{F}} > \omega_1$	4.4.1 and 4.4.8
Rainbow coloring o^*	CA_2	3.4.9
Sixth Tukey type	CA_2	3.4.20
$\omega_1 \nrightarrow (\omega_1, \omega + 2)_2^2$	CA_2	3.3.5
Non productivity of ccc	CA_2	3.4.10
Suslin towers	CA_2	3.4.22
S-spaces	CA_2	3.4.29, 3.4.32 and 3.4.46
Suslin lower semi-lattices	CA_2	3.2.26
Failure of $BA(\omega_1)$	CA_2	3.4.54
ccc destructible 2-bounded coloring	CA_3	3.3.9
Uncountable metric space without uncountable monotone subspaces	CA	3.5.7
Coherent Suslin Tree	$FCA(part)$	3.2.18
Full Suslin Tree	FCA	3.2.22
Entangled sets	FCA	3.2.33
Independent coherent family of functions	FCA	3.1.88

In addition to applications, the main contributions of this work are listed below. All of the undefined notions will be defined throught the thesis.

Result	Extra set-theoretic assumption	Result number
There are no strongly donut-separable gaps	Either PFA or CH	3.1.39 and 3.1.53
MA is independent from the statement "There is a strongly donut-separable gap"	-	3.1.54
There are no destructible gaps	$\mathfrak{m}_{\mathcal{F}} > \omega_1$	3.1.82
There are no Suslin trees	$\mathfrak{m}_{\mathcal{F}} > \omega_1$	3.2.24
$\mathfrak{c} = \mathfrak{m}_{\mathcal{F}}^n > \omega_1$ is consistent with an arbitrarily large continuum	-	4.1.8
Consistency of the existence of a construction scheme $\mathcal{F}$ which is $n+1$-capturing, $\mathcal{P}$-n-capturing, but not $\mathcal{P}'$-$n+1$-capturing	-	4.3.9
There are no 2-capturing construction schemes	Either $\mathfrak{m} > \omega_1$ or PID	4.3.4 and 5.3.12
Full capturing with partitions axiom ($FCA(part)$)	$\lozenge$	5.2.31

Most of the results presented here were obtained in a joint work with Osvaldo Guzmán and Stevo Todorčević. Part of them appear in [24] which has already been submitted.

Chapter 1

Preliminaries and Notation

We devote this chapter to fix most of the preliminaries, notation and terminology that will be used throughout the rest of this work. Broadly speaking, the results appearing in this text can be separated into two categories. The ones regarding the development of the theory of *construction schemes* and the ones regarding applications of this theory to set theory and topology. Of course, familiarity with the basics of these two branches of mathematics is assumed. Nothing more is asked for the reader solely interested in applications. However, familiarity with the forcing method is mandatory for the reader interested in learning more about the development of the theory. For set theory, the texts Halbeisen [44], Jech [54] and Kunen [64] provide, in particular, all the prerequisites needed. The texts Engelking [34], Nagata [82] and Willard [131] do the same for the topological background.

1.1 Set-Theoretic Notation

Ordinals are usually denoted by the first lower case Greek letters $\alpha, \beta, \gamma, \delta, \ldots$, whereas the middle letters $\kappa, \lambda, \mu, \ldots$ are reserved for cardinals (commonly infinite). The set of natural numbers is denoted by ω and its elements are mainly denoted by the lowercase letters in the English alphabet. ω_1 stands for the first uncountable ordinal. The set of all limit ordinals smaller than ω_1 is denoted by Lim.

We denote the power set of X by $\mathscr{P}(X)$. The cardinality of X is denoted by $|X|$. Given a cardinal κ, we let $[X]^\kappa$ be the set of all subsets of X of size κ and $[X]^{<\kappa}$ denotes $\bigcup_{\alpha<\kappa}[X]^\alpha$. The set $[X]^{\leqslant\kappa}$ is defined in a similar way. By $\mathrm{FIN}(X)$ we mean $[X]^{<\omega}\setminus\{\emptyset\}$. $\mathrm{FIN}(\omega)$ is just denoted as FIN. We write $X \subseteq^* Y$ if $X\setminus Y$ is finite and $X =^* Y$ if $X \subseteq^* Y$ and $Y \subseteq^* X$. Following this notation, we write $X \subsetneq^* Y$ whenever $X \subseteq^* Y$ but $X \neq^* Y$.

Definition 1.1.1 (Δ-system)**.** A family $\mathcal{D}$ is called a Δ-*system* with root R if $|\mathcal{D}| \geqslant 2$ and $X \cap Y = R$ whenever $X, Y \in \mathcal{D}$ are different.

Let $f : X \longrightarrow Y$ be a function. As usual, for $x \in X$ we denote the image of x under f as $f(x)$. For $A \subseteq X$ and $B \subseteq Y$ the *direct image* of A under f, that is, $\{f(x) : x \in A\}$, is denoted as $f[A]$ and the *inverse image* of B, that is, $\{x \in X : f(x) \in B\}$ is

denoted by $f^{-1}[B]$. Sometimes we write $dom(f)$ instead of X $im(f)$ instead of $f[X]$. Whenever we write $g; X \longrightarrow Y$ we mean that g is a function with $dom(g) \subseteq X$ and $im(g) \subseteq Y$. The set of all functions from X to Y is denoted by X^Y. For an ordinal β we define $X^{<\beta}$ and $X^{\leqslant \beta}$ as $\bigcup_{\alpha < \kappa} X^\alpha$ and $X^{<\beta} \cup X^\beta$ respectively.

1.2   Order-Theoretic Notation

Let $(X, <)$ be a partial order and $x, y \in X$. We say x and y are *comparable* if either $x \leqslant y$ or $y \leqslant x$. We say x and y are *compatible* and write it as $x \parallel y$ if there is $r \in X$ with both $r \leqslant x$ and $r \leqslant y$. The *incompatibility* relation between x and y is written as $x \perp y$. We define the following sets:

$$(-\infty, x)_X = \{z \in X \,:\, z < x\} \qquad (x, y)_X = (x, \infty) \cap (-\infty, y)$$
$$(-\infty, x]_X = \{z \in X \,:\, z \leqslant x\} \qquad (x, y]_X = (x, \infty) \cap (-\infty, y]$$
$$(x, \infty)_X = \{z \in X \,:\, x < z\} \qquad [x, y)_X = [x, \infty) \cap (-\infty, y)$$
$$[x, \infty)_X = \{z \in X \,:\, x \leqslant z\} \qquad [x, y]_X = [x, \infty) \cap (-\infty, y]$$

We say that $A \subseteq X$ is an *interval* in X whenever $(x, y) \subseteq A$ for all $x, y \in A$. A is an *initial segment of* X if $(-\infty, x) \subseteq A$ for every $x \in A$. A is a *final segment of* X if $(x, \infty) \subseteq A$ whenever $x \in A$. A is cofinal in X if for any $[x, \infty) \cap A$ is non-empty for all $x \in X$. A is a *chain* if any two elements of A are comparable. On the other hand, if any two elements of A are incomparable, we call A a *pie*. Finally, A is an *antichain* in X if any two elements in A are incompatible.

Let $B \subseteq X$ and $x \in B$. x is said to be a *minimal* element of B if there is no $y \in B$ with $y < x$. Furthermore, if $x \leqslant y$ for any $y \in B$ we say that x is the *minimum of* B and denote it as $\min(B)$. Analogously, x is said to be a maximal element of B if there is no $y \in B$ with $y > x$. Moreover, $x \geqslant y$ for any $y \in B$ we call x the *maximum* of B and denote it as $\max(B)$. An element $z \in X$ is an *upper bound* (resp. *lower bound*) of B if $z = \max(B \cup \{z\})$ (resp. $z = \min(B \cup \{z\})$). In case the set of upper bounds of B has a minimum, it is denoted as $\sup B$. In a similar way, if the set of lower bounds of B has a maximum, it is denoted as $\inf B$. Lastly, for an element $x \in X$ we denote by

$$suc(x) = \{y \in (x, \infty) \,:\, y \text{ is a minimal element of } (x, \infty)\,\},$$

$$pred(x) = \{y \in (-\infty, x) \,:\, y \text{ is a maximal element of } (-\infty, x)\,\}.$$

Note that $y \in suc(x)$ if $x < y$ and there is no $z \in X$ with $x < z < y$. Analogously $y \in pred(x)$ if $y < x$ and there is no $z \in X$ such that $y < z < x$. Let $A, B \subseteq X$ be arbitrary subsets of X. We say $A < B$ if B is non-empty and $a < b$ for all $a \in A$ and $b \in B$. We say that B *end-extends* A (or A is *end-extended* by B) and write it as $A \sqsubseteq B$ if A is an initial segment of B.

Definition 1.2.1 (Root-tail-tail Δ-system)**.** A Δ-system $\mathcal{D} \subseteq \mathscr{P}$ with root R is said to be *root-tail-tail* if $R \sqsubseteq A$ and either $A \backslash R < B \backslash R$ or $B \backslash R < A \backslash R$ for any two distinct $A, B \in \mathcal{D}$. In this same situation, if $\mathcal{D}$ is indexed by some ordinal γ as, say, $\langle D_\alpha \rangle_{\alpha < \gamma}$ we will make the implicit assumption that $D_\alpha \backslash R < D_\beta \backslash R$ whenever $\alpha < \beta$.

X is said to be *well-founded* if any non-empty subset of X has a minimal element. In this case, there is a unique function $rank : X \longrightarrow Ord$ satisfying:

$$rank(x) = \sup(\, rank(y) + 1 \,:\, y < x \,)$$

Note that this function is order preserving and its image is an ordinal. For each ordinal α, we define the *level* α *of X* as $X_\alpha = \{x \in X \,:\, rank(x) = \alpha\}$ and the *height of X* as $Ht(X) = \min(\, \alpha \in Ord \,:\, X_\alpha = \emptyset \,) = rank[X]$. X is said to be a *well-order* if any non-empty subset of X has a minimum. In this case, there is a unique ordinal β and a unique order preserving function $\phi : \alpha \longrightarrow X$ called *the enumeration of X*. We call β the order type of X and denote it as $ot(X)$.

Very important notational remark In the previous situation, following what some people may call abuse of notation, we identify X with ϕ. In this way, $X(\alpha)$ denotes $\phi(\alpha)$ and $X^{-1}(x)$ denotes $\phi^{-1}(x)$ whenever $\alpha \in \beta$ and $x \in X$. In the same way, $X[S]$ denotes $\phi[S]$ and $X^{-1}[Y]$ denotes $\phi^{-1}[Y]$ for all $S \subseteq \alpha$ and $Y \subseteq X$.

Definition 1.2.2 (Product order and lexicographical order)**.** Whenever $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ are two partial orders, we define the following two partial orders over the product $X \times Y$; the *order product* $\leqslant$ given by $(x, y) \leqslant (w, z)$ if $x \leqslant_X w$ and $y \leqslant_Y z$ and the *lexicographical product* $\leqslant_{lex}$ given by $(x, y) \leqslant_{lex} (w, z)$ if either $x <_X w$ or $x = w$ and $y \leqslant_Y z$.

Definition 1.2.3 (Lower semi-lattice)**.** $(X, \leqslant, \wedge)$ is said to be a *lower semi-lattice* if $(X, \leqslant)$ is a partial order and $\wedge : X^2 \longrightarrow X$ is such that for all $x, y \in X$, $\inf(x, y)$ exists and it is equal to $x \wedge y$.

1.3 Boolean algebras and filters

Boolean algebras are one of the central objects of study in modern set theory and set-theoretic topology. Strictly speaking, all the results appearing in this thesis can be presented by avoiding any explicit mention to these objects. However, I think that, at least in this case, they provide a perfect setting for the motivation and analysis of some of the main topics which will be discussed eventually. We will briefly review some of the basic definitions and theorems. A detailed treatment of the subject can be found in Koppelberg [62].

Definition 1.3.1. A Boolean algebra is a structure $(\mathbb{A}, +, \cdot, -, \mathbb{0}, \mathbb{1})$ with two binary operations[1] $+$ and $\cdot$, a unary operation $-$, and two (distinct) distinguished elements $\mathbb{0}$ and $\mathbb{1}$ such that for all $x, y, z \in A$ the following conditions hold:

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[1]Here we decided to adopt the "algebraic" notation for the operations in Boolean algebras. Many authors prefer the usage of the symbols $\vee$, $\wedge$, and $\neg$ to refer to these operations (see Givant and Halmos [41]).

- $x + (y + z) = (x + y) + z,$
- $x \cdot (y \cdot z) = (x \cdot y) \cdot z,$
- $x + y = y + x,$
- $x \cdot y = y \cdot x,$
- $x \cdot (y + z) = x \cdot y + x \cdot z,$
- $x + (y \cdot z) = (x + y) \cdot (x + z),$
- $x + y \cdot x = x,$
- $x \cdot (y + x) = x,$
- $x + (-x) = \mathbb{1},$
- $x \cdot (-x) = \mathbb{0}.$

The most tipical example of a Boolean algebra is $(\mathscr{P}(X), \cup, \cap, {}^c, \emptyset, X)$ where ^{c} represents the complementation with respect to X.

Any Boolean algebra has an associated partial order given by

$$x \leqslant y \text{ if } x \cdot y = x(\text{equivalently } x + y = y).$$

In this order, $\mathbb{0}$ and $\mathbb{1}$ are the minimum and maximum of $\mathbb{A}$, respectively. In addition, Boolean operations can be completely recovered by the order in the sense that $\sup\{x, y\} = x + y$ and $\inf\{x, y\} = x \cdot y$. In the case of $\mathscr{P}(X)$ the order induced by the boolean operations coincide with " $\subseteq$".

Definition 1.3.2. Let $\mathbb{A}$ be a Boolean algebra and let $\mathcal{F}, \mathcal{I} \subseteq \mathbb{A}$. We say that

$\mathcal{F}$ is a *filter* if:

- $\mathbb{1} \in \mathcal{F}$ and $\mathbb{0} \notin \mathcal{F}$,
- $x \cdot y \in \mathcal{F}$ whenever $x, y \in \mathcal{F}$,
- If $x \in \mathcal{F}$ and $y \geqslant x$ then $y \in \mathcal{F}$.

$\mathcal{I}$ is an *ideal* if:

- $\mathbb{0} \in \mathcal{I}$ and $\mathbb{1} \notin \mathcal{I}$,
- $x + y \in \mathcal{I}$ whenever $x, y \in \mathcal{I}$,
- If $x \in \mathcal{I}$ and $y \leqslant x$ then $y \in \mathcal{I}$.

A maximal filter is called an ultrafilter, and a maximal ideal is called a prime ideal.

In this text, we will work mainly with filters over Boolean algebras of the form $\mathscr{P}(X)$. In this context, a filter $\mathcal{F}$ over $\mathscr{P}(X)$ (or simply a filter over X) is a non-empty family of non-empty subsets of X which is closed upwards and under finite intersections. We call such $\mathcal{F}$ *principal* if there is $A \subseteq X$ for which $\mathcal{F} = \{B \subseteq X : A \subseteq B\}$. If there is no such A, $\mathcal{F}$ is called non-principal. Analogously, an ideal $\mathcal{I}$ over $\mathscr{P}(X)$ (or simply an ideal over X) is a family of proper subsets of X that is closed downwards and under finite unions.

Let $\mathcal{F}$ be a filter over some set X. We define the *family of positive sets with respect to $\mathcal{F}$*, that is, $\mathcal{F}^+$, as the set of all $A \subseteq X$ which intersect each member of $\mathcal{F}$. In the same way, if $\mathcal{I}$ is an ideal in some set X, we define $\mathcal{I}^+$ as $\mathscr{P}(X) \backslash \mathcal{I}$.

Definition 1.3.3 (Quotient algebras)**.** Let $\mathbb{A}$ be a Boolean algebra and let $\mathcal{I} \subseteq \mathbb{A}$ be an ideal. $\mathcal{I}$ naturally induces an equivalence relation "$\sim$" on $\mathbb{A}$ given by

$$x \sim y \text{ if and only if } x \cdot (-y) + y \cdot (-x) \in \mathcal{I}.$$

We denote the equivalence class of x under such relation as $[x]$. The set of equivalence classes is denoted as $\mathbb{A}/\mathcal{I}$. The operations on $\mathbb{A}/\mathcal{I}$ given by $-[x] = [x], [x]+[y] = [x+y],$

and $[x] \cdot [y] = [x \cdot y]$ are well-defined. Furthermore, $(\mathbb{A}/\mathcal{I}, +, \cdot, -, [\mathbb{0}], [\mathbb{1}])$ is a Boolean algebra.

The Boolean algebra in which we will focus most of our attention is $\mathscr{P}(\omega)/\text{FIN}$. Note that in this case, if $X, Y \in \mathscr{P}(\omega)$ then $[X] \leqslant [Y]$ if and only if $X \subseteq^* Y$.

1.4   Forcing

Throughout this thesis, we will force downwards. Thus, a *forcing notion* (or simply a *forcing*) is triplet $(\mathbb{P}, \leqslant, \mathbb{1}_\mathbb{P})$ so that $(\mathbb{P}, \leqslant)$ is a partial order with $\mathbb{1}_\mathbb{P}$ as its maximum. Usually, we will denote $(\mathbb{P}, \leqslant, \mathbb{1}_\mathbb{P})$ simply as $\mathbb{P}$ unless there is any risk of confusion. A subset D of $\mathbb{P}$ is said to be *dense* if for any $q \in \mathbb{P}$ there is $p \in D$ so that $p \leqslant q$. Lastly, we call a subset G of $\mathbb{P}$ a filter if G is non-empty, it is closed upwards and for any $p, q \in G$ there is $r \in G$ so that $r \leqslant p, q$.

Definition 1.4.1 (Cohen forcing). Given κ an infinite cardinal, we define the κ-*Cohen forcing* as

$$\mathbb{C}_\kappa = \{\, p; \kappa \longrightarrow 2 \,:\, p \text{ is finite}\}$$

and order with the reverse inclusion. The forcing C_ω is simply called *Cohen forcing* and denoted as $\mathbb{C}$.

Let V be a transitive model of ZFC and $\mathbb{P} \in V$. We say that a filter G is $\mathbb{P}$-*generic over V* if G intersects any dense set of $\mathbb{P}$ lying in V. In this situation, there is a minimal transitive model of ZFC which extends V and has G as an element. This model is denoted as $V[G]$.

The forcing relation is denoted as $\Vdash$. Given a formula ϕ, we write $\mathbb{P} \Vdash$ "ϕ" instead of $\mathbb{1}_\mathbb{P} \Vdash$ "ϕ". We adopt the convention of denoting $\mathbb{P}$-names with dots above them. However, if X is an element of the universe V, we will make an abuse of notation and denote the canonical $\mathbb{P}$-name of X simply as X. Whenever $\dot{Y}$ is a $\mathbb{P}$-name and G is a $\mathbb{P}$-generic filter over V, we denote the interpretation of $\dot{Y}$ with respect to G as $\dot{Y}^G$. Given an ordinal γ, *finite support iterations* of length γ are denoted as

$$(\langle \mathbb{P}_\xi \rangle_{\xi \leqslant \gamma}, \langle \dot{\mathbb{Q}}_\xi \rangle_{\xi < \gamma}).$$

For us, elements $p \in \mathbb{P}_\gamma$ are finite partial functions with domain contained in γ.

We now define some classes of forcing notions which will be important in further discussions.

Definition 1.4.2 (*ccc* forcings). A forcing $\mathbb{P}$ is said to be *ccc* if any antichain in $\mathbb{P}$ is at most countable.

Definition 1.4.3 (Knaster forcings). Let $2 \leqslant n \in \omega$. Given a forcing $\mathbb{P}$, we say that $\mathcal{B} \subseteq \mathbb{P}$ is *n-linked* if for any $p_0, \ldots, p_{n-1} \in \mathcal{B}$ there is $p \in \mathbb{P}$ (not necessarily in $\mathcal{B}$) so that $p \leqslant p_i$ for each $i < n$. $\mathbb{P}$ is said to be *n-Knaster* if for any $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$ there is $\mathcal{B} \in [\mathcal{A}]^{\omega_1}$ which is n-linked. 2-Knaster forcings are simply called *Knaster*.

Definition 1.4.4 (σ-centered forcings.)**.** Given a forcing $\mathbb{P}$, a subset $\mathcal{C}$ of $\mathbb{P}$ is said to be *centered* if it is n-linked for each $2 \leqslant n \in \omega$. We say that $\mathbb{P}$ is σ-*centered* if $\mathbb{P} = \cup_{l \in \omega} \mathcal{C}_l$ where $\mathcal{C}_l$ is centered for each l.

Definition 1.4.5 (Precaliber)**.** Let κ be an infinite cardinal. We say that a forcing $\mathbb{P}$ has *precaliber* κ if for each sequence $\langle p_\alpha \rangle_{\alpha \in \mathbb{P}}$ there is $X \in [\kappa]^\kappa$ for which $\{p_\alpha : \alpha \in X\}$ is centered.

Remark 1.4.6. Any σ-centered forcing has precaliber ω_1, and each forcing with precaliber ω_1 is n-Knaster for each $2 \leqslant n \in \omega$. Furthermore, any Knaster forcing is *ccc*.

The following theorem will be frequently used.

Theorem 1.4.7. *Let $\mathbb{P}$ be an uncountable ccc forcing. Then there is $p \in \mathbb{P}$ which forces the generic filter to be uncountable.*

Given a forcing $\mathbb{P}$, we define the *Martin's number of* $\mathbb{P}$, namely $\mathfrak{m}(\mathbb{P})$, as the minimal cardinal κ for which there is a family $\mathcal{D}$ of size κ of dense sets of $\mathbb{P}$ so that there is no filter G intersecting each member of $\mathcal{D}$. To learn more about Martin's numbers, see [12] and [111]. Regarding this notion, we define the following cardinal invariants:

$$\mathfrak{m} = \min(\mathfrak{m}(\mathbb{P}) : \mathbb{P} \text{ is ccc}).$$

$$\mathfrak{m}_{K_n} = \min(\mathfrak{m}(\mathbb{P}) : \mathbb{P} \text{ is } n\text{-Knaster}).$$

Theorem 1.4.8. *Suppose that $\mathbb{P}$ is an uncountable ccc forcing so that $\mathfrak{m}(\mathbb{P}) > \omega_1$. Then $\mathbb{P}$ has an uncountable filter.*

Two well-known axioms regarding the Martin's numbers of certain classes of forcing notions will appear in some discussions, namely MA and PFA. We will state them bellow for the sake of completeness. The notion of *property* will not be defined here. However, the reader may find a full treatment of this subject in [95].

Martin's Axiom [MA]: $\mathfrak{m} = \mathfrak{c} > \omega_1$.

Martin's Axiom for n-Knaster forcings [MA(K_n)]: $\mathfrak{m}_{K_n} = \mathfrak{c} > \omega_1$.

Proper Forcing Axiom [PFA]: $\mathfrak{m}(\mathbb{P}) > \omega_1$ for any proper forcing notion $\mathbb{P}$.

1.5 P-ideal Dichotomy

Let $\mathcal{I}$ be an ideal over a set X. We say that $\mathcal{I}$ is a *P-ideal* if for each countable $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ such that $A \subseteq^* B$ for any $A \in \mathcal{A}$. Any such set B is called a *pseudo-union* of $\mathcal{A}$. The *P*-ideal dichotomy (PID) is a well-known consequence of PFA regarding *P*-ideals. Stevo Todorčević introduced it in its current form in [123]. There he showed that PID is consistent with CH, although PFA is not. A less general version of PID was previously formulated by Uri Abraham and Todorčević in [4].

Definition 1.5.1. Let $\mathcal{I}$ be an ideal over a set X. We define the *orthogonal ideal of* $\mathcal{I}$, namely $\mathcal{I}^\perp$, as the set of all $A \subseteq X$ such that $A \cap B =^* \emptyset$ for any $B \in \mathcal{I}$.

P**-ideal Dichotomy [PID]:** Let X be a set and $\mathcal{I}$ be a P-ideal over X such that $[X]^{<\omega} \subseteq \mathcal{I} \subseteq [X]^{\leqslant\omega}$. Then one of the two following conditions hold:

- There is $Y \in [X]^{\omega_1}$ so that $[Y]^\omega \subseteq \mathcal{I}$.

- There is $\{Z_n : n \in \omega\} \subseteq \mathcal{I}^\perp$ for which $X = \bigcup_{n\in\omega} Z_n$.

1.6   Diamond principle

Undoubtedly, the most important axiom regarding this work is Jensen's $\Diamond$-principle. This principle was introduced by Ronald Jensen in [56]. In there, he showed that this principle holds in the constructible universe L. We will recall the version of the $\Diamond$-principle that we will be using.

A $\Diamond$-sequence (over ω_1) is a sequence $\langle D_\alpha \rangle_{\alpha\in\mathrm{Lim}}$ of countable subsets of ω_1 for which the following properties hold:

(a) For each $\alpha \in \mathrm{LIM}$, $D_\alpha \subseteq \alpha$.

(b) Given $S \subseteq \omega_1$, the set $\{\alpha \in \mathrm{Lim} : S \cap \alpha = D_\alpha\}$ is stationary.

Jensen's $\Diamond$-principle [$\Diamond$]: There is a $\Diamond$-sequence.

Chapter 2

Ordinal metrics and construction schemes

In this chapter we introduce ordinal metrics and construction schemes as well as study the relationship between them.

Stevo Todorčević implicitly introduced ordinal metrics in [120][1], as a tool for showing the existence of a coloring of the pairs of countable ordinals with uncountably many colors so that every uncountable set contains pairs of every color. In [112], he introduced construction schemes as a tool for building uncountable structures by finite approximations. Broadly speaking, construction schemes provide blueprints for the domain of such approximations, as well as a procedure to recursively improve them through the use of finite amalgamations. This concept can be viewed as a generalisation of the so-called $(\omega, 1)$-morasses as defined by Ronald Jensen (see [26]) and simplified by Daniel Velleman in [128]. The main difference between these two notions is that with morasses we always amalgamate *two* objects when improving approximations, whereas in construction schemes the number of amalgamations allowed may vary.

The relationship between ordinal metrics and construction schemes has already been highlighted in [112]. It is worth pointing out that Charles Morgan presented in [81] a detailed analysis of the relation between ordinal metrics and gap-1 morasses. Readers who want to know more about morasses and their applications may look at [26], [33], [60], [92], [101], [128], [129], and [130].

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[1]In Section 2 of such paper, it is proved in (2.3) that the function ρ defined immediately before satisfies the properties of what we now call an ordinal metric.

2.1   Ordinal metrics

For the rest of this chapter, X stands for a non-empty set of ordinals.

Definition 2.1.1 (Closure operation). Let $\rho : X^2 \longrightarrow \omega$ be an arbitrary function. For each $F \in \mathrm{FIN}(X)$ and $k \in \omega$ we define the *k-closure* of F as:

$$(F)_k = \{\, \alpha \in X \ : \ \exists \beta \in F \backslash \alpha \, (\rho(\alpha, \beta) \leqslant k) \,\}$$
$$(F)_k^- = (F)_k \cap \max(F) = (F)_k \backslash \{\max(F)\}$$

The *diameter* of F is defined as

$$\rho^F = \max(\, \rho(\alpha, \beta) \ : \ \alpha, \beta \in F \,).$$

F is said to be *k-closed* whenever $F = (F)_k$. Moreover, if F is ρ^F-closed we just say that F is *closed*. For $\alpha \in X$, we will write $(\alpha)_k$ and $(\alpha)_k^-$ instead of $(\{\alpha\})_k$ and $(\{\alpha\})_k^-$ respectively.

Remark 2.1.2. Both the k-closure and the diameter are monotone operations. That is, if $F \subseteq G$ then $(F)_k \subseteq (G)_k$ and $\rho^F \leqslant \rho^G$.

Following the convention established in the preliminaries, if $(\alpha)_k = \{\beta_0, \ldots, \beta_n\}$ with $\beta_0 < \ldots < \beta_n$, we write $(\alpha)_k(i) = \beta_i$ for each $i \leqslant n$. This notation will be used throughout this thesis.

Definition 2.1.3 (*k*-cardinality). Let ρ be as in the previous definition. Given $\alpha \in X$ and $k \in \omega$ we define the *k-cardinality* of α as

$$\|\alpha\|_k = |(\alpha)_k^-|.$$

Now we define ordinal metrics following the same approach (although slightly different notation) as in [110]. In practice, we will only be interested in ordinal metrics whose domain is ω_1^2. However, such metrics are constructed via recursion. For that reason, it is useful to present the definition in this generality.

Definition 2.1.4 (Ordinal metric). [2] We say that $\rho : X^2 \longrightarrow \omega$ is an *ordinal metric* (over X) if:

(a) $\forall \alpha, \beta \in X \, (\, \rho(\alpha, \beta) = 0 \ \textit{if and only if} \ \alpha = \beta \,)$,

(b) $\forall \alpha, \beta \in X \, (\, \rho(\alpha, \beta) = \rho(\beta, \alpha) \,)$,

(c) $\forall \alpha, \beta, \gamma \in X \, (\, \alpha \leqslant \beta, \gamma \to \rho(\alpha, \beta) \leqslant \max(\rho(\alpha, \gamma), \rho(\beta, \gamma)) \,)$,

(d) $\forall \beta \in X \, \forall k \in \omega \, (\, \|\beta\|_k < \omega \,)$.

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[2]Some authors prefer to call ordinal metrics "T-functions" or "ρ-functions".

As the reader may note, the previous definition resembles that of an ultra-metric. However, one of the triangle inequalities is missing. In this way, one may interpret $(\beta)_k$ as the ball of radius k centered on β intersected with $\beta + 1$. In the theory of metric spaces, we use the metric to approximate or locate points in a given space. To do that, we study the behaviour of balls centered on a given point as they become smaller. Ordinal metrics tend to work in a different way. Here, we want to construct a structure whose elements are parameterized by the points of the domain of our ordinal metric, say X. In order to do that, we make approximations of such structure by analyzing the interaction between the elements $\beta \in X$ and the points in $(\beta)_k$ as k grows larger. For that reason it may not be always suitable to carry the intuition of metric spaces into the context of ordinal metrics, but rather use the word "metric" to help us remember the main aspects of the definition. With the following proposition we show the difference between these two notions in a formal sense.

Proposition 2.1.5. *No ordinal metric over ω_1 is a metric.*

Proof. Suppose ρ is as previously mentioned and let M be a countable elementary submodel of $H(\omega_2)$ having ρ as an element. We will show that ρ does not satisfy the triangle inequality of metric spaces. Take $\gamma \in \omega_1 \backslash M$, consider an arbitrary $\alpha \in M \cap \omega_1$ and define k as $\rho(\alpha, \gamma)$. Using elementarity we can find $\beta \in M \cap \omega_1$ above $\max((\gamma)_{2k} \cap M)$ for which $k = \rho(\alpha, \beta)$. By definition of the k-closure, we have that $\rho(\beta, \gamma) > 2k = \rho(\alpha, \beta) + \rho(\alpha, \gamma)$. This finishes the proof. ∎

If ρ is an ordinal metric over X and $\alpha \in X$ then $(-\infty, \alpha]_X = \bigcup_{k \in \omega}(\alpha)_k$. Due to the property (d) of Definition 2.1.4 this set must be countable. Hence, the existence of such ρ implies that $ot(X)$ is at most ω_1.

Remark 2.1.6. If ρ is an ordinal metric over X and $F \in \mathrm{FIN}(X)$ then $F \subseteq (F)_k$ for each $k \in \omega$. Furthermore, as the k-closure operation does not add points above the maximum of the set to which it is applied, it follows that $\max((F)_k) = \max(F)$ for any such F.

The following proposition summarizes the most basic properties regarding the closure of sets in the context of ordinal metrics.

Proposition 2.1.7. *Let $\rho : X^2 \longrightarrow \omega$ be and ordinal metric, $F \in FIN(X)$ and $k \geqslant \rho^F$. Then:*

(1) $\rho^F = \max(\rho(\alpha, \max(F)) : \alpha \in F)$.

(2) $(F)_k = \{ \alpha \leqslant \max(F) : \rho(\alpha, \max(F)) \leqslant k \} = (\max(F))_k$.

(3) *If $\beta \in F$, then $(F)_k \cap (\beta + 1) = (F \cap (\beta + 1))_k$.*

(4) $(F)_k$ *is k-closed and $\rho^{(F)_k} \leqslant k$, so in particular $(F)_k$ is closed. Furthermore, if $k = \rho^F$, then $\rho^{(F)_k} = k$.*

(5) F *is closed if and only if $F = (\max(F))_{\rho^F}$.*

(6) *If $G \in FIN(X)$ is k-closed, then $F \cap G \sqsubseteq F$.*

Proof. The points (1) and (2) are direct consequences of the property (c) of Definition 2.1.4 and the points (4) and (5) are direct consequences of the point (1) and (2). Thus, we will only prove the points (3) and (6).

Proof of (3). It is clear that $(F \cap (\beta + 1))_k \subseteq (F)_k \cap (\beta + 1)$ so we will only prove that $(F)_k \cap (\beta + 1) \subseteq (F \cap (\beta + 1))_k$. For this take an arbitrary $\alpha \in (F)_k \cap (\beta + 1)$. By the point (2) we have that $\rho(\alpha, \max(F)) \leqslant k$. Using the property (c) of Definition 2.1.4 we can conclude that

$$\rho(\alpha, \beta) \leqslant \max(\rho(\alpha, \max(F)), \rho(\beta, \max(F))) \leqslant k.$$

Hence $\alpha \in (F \cap (\beta + 1))_k$ and we are done. $\qquad\qquad\qquad\qquad\qquad\square$

Proof of (6). Fix $\beta \in F \cap G$ and let $\alpha \in F$ with $\alpha \leqslant \beta$ be arbitrary. Then $\rho(\alpha, \beta) \leqslant \rho^F \leqslant k$. As G is k-closed, $\alpha \in G$. In this way $\alpha \in F \cap G$. This finishes the proof. $\quad\square$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\blacksquare$

In this way, in order to compute ρ^F and $(F)_k$, we only need to use the maximum of F.

Corollary 2.1.8. *Let ρ be an ordinal metric over X and $F, G \in FIN(X)$ with $F \sqsubseteq G$. If G is closed then so is F.*

Proof. As G is closed then $G = (G)_{\rho^G}$. Let $\beta = \max F$. Since $F \sqsubseteq G$ then $\beta \in G$ and $F = G \cap (\beta + 1)$. By the point (3) of Proposition 2.1.7 we have that

$$(F)_{\rho^G} = (G \cap (\beta + 1))_{\rho^G} = (G)_{\rho^G} \cap (\beta + 1) = G \cap (\beta + 1) = F.$$

This means that F is ρ^G-closed. But $F \subseteq G$ so $\rho^F \leqslant \rho^G$. Consequently F is closed. $\quad\blacksquare$

We are interested in ordinal metrics which belong to three particular classes. In the rest of this section we will study them. But first, we need the following definition.

Definition 2.1.9. Let $F \in \mathrm{FIN}(X)$. We say that F is *maximally closed* if it is closed and there is no closed $G \in \mathrm{FIN}(X)$ with $\rho^G = \rho^F$ such that $F \subsetneq G$.

Definition 2.1.10 (Locally finite metric)**.** Let $\rho : X^2 \longrightarrow \omega$ be an ordinal metric. We say that ρ is *locally finite* if

$$\sup(\,|F| \ : \ F \in \mathrm{FIN}(X), F \text{ is closed and } \rho^F \leqslant k\,)$$

is finite for every $k \in \omega$.

Remark 2.1.11. If ρ is locally finite and $F \in \mathrm{FIN}(X)$ then there is a maximally closed set $G \in \mathrm{FIN}(X)$ such that $F \subseteq G$ an $\rho^F = \rho^G$. In fact, due to the point (6) of Proposition 2.1.7 we can conclude that $(F)_{\rho^F} \sqsubseteq G$ for any such G. In particular, any closed set can be end-extended to a maximally closed set of the same diameter.

The following result is an immediate consequence of the previous remark.

Proposition 2.1.12. *Let $\rho : X^2 \longrightarrow \omega$ be a locally finite ordinal metric. For each $F \in FIN(X)$ we have that*

$$\rho^F = \min(\,\rho^G \,:\, G \text{ is maximally closed and } F \subseteq G\,)$$
$$= \min(\,\rho^G \,:\, G \text{ is maximally closed and } F \sqsubseteq G\,)$$

Proposition 2.1.13. *Let $\rho : X^2 \longrightarrow \omega$ be a locally finite ordinal metric. For all $k \in \omega$ and every $A \in [X]^\omega$ there exist $\alpha, \beta \in A$ such that $\rho(\alpha, \beta) > k$.*

Proof. Let k and A be as in the hypotheses. Suppose towards a contradiction that $\rho(\alpha, \beta) \leqslant k$ for all $\alpha, \beta \in A$. Then $\rho^{A[n]} \leqslant k$ for each $n \in \omega$. Due to the point (6) of Proposition 2.1.7 we conclude that $\langle (A[n])_k \rangle_{k \in \omega}$ is a strictly increasing sequence of closed sets all of whose diameter is at most k. This contradicts the fact that ρ is locally finite. So we are done. ∎

Ordinal metrics satisfying the conclusion of the previous proposition deserve a special name.

Definition 2.1.14 (unbounded metric)**.** Let $\rho : X^2 \longrightarrow \omega$ be an ordinal metric. We say that ρ is *unbounded* if for all $k \in \omega$ and every $A \in [X]^\omega$ there are $\alpha, \beta \in A$ with $\rho(\alpha, \beta) > k$.

The following proposition can be found in [110] (Lemma 3.4.11, page 73). The proof that we present here avoids the use of ultrafilter quantifiers.

Proposition 2.1.15. *Let $\rho : X^2 \longrightarrow \omega$ be an unbounded ordinal metric and $k \in \omega$. For all $l \in \omega$ and each infinite $A \subseteq [X]^l$ family of pairwise disjoint sets there is $B \in [A]^\omega$ such that for all distinct $a, b \in B$ and every $\alpha \in a$ and $\beta \in b$, we have that $\rho(\alpha, \beta) > k$.*

Proof. Let l and A be as in the hypotheses. Enumerate A as $\langle a_n \rangle_{n \in \omega}$. For any $i < l$ there are two particular colorings that we may consider. The first one is the coloring $c_i : [\langle a_n(i) \rangle_{n \in \omega}]^2 \longrightarrow 2$ given by $c_i(a_m(i), a_n(i)) = 0$ if and only if $m < n$ and $a_m(i) < a_n(i)$. The second one is the coloring $o_i : [\langle a_n(i) \rangle_{n \in \omega}]^2 \longrightarrow 2$ given by $o_i(a_m(i), a_n(i)) = 0$ if and only if $\rho(a_m(i), a_n(i)) > k$. Since there are no infinite decreasing sequences of ordinals and ρ is an unbounded metric then all infinite monochromatic sets with respect to these two coloring choose color 0. In this way, by applying Ramsey's Theorem multiple (but finitely many) times, we can get a set $B \in [A]^\omega$ enumerated as $\langle b_n \rangle_{n \in \omega}$ satisfying the following conditions for each $i < l$ and all $m < n \in \omega$:

- $b_m(i) < b_n(i)$,

- $\rho(b_m(i), b_n(i)) > k$.

and such that for all distinct $i, j < l$ either:

(A)	(B)
For all $m < n \in \omega$, $\rho(b_m(i), b_n(j)) \leqslant k$.	For all $m < n \in \omega$, $\rho(b_m(i), b_n(j)) > k$.

The proof wil be finished if we are able show that (B) holds for all distinct $i, j < l$. We divide the proof of this fact into three cases. In each of them, it is enough to show that there are $m < n$ for which $\rho(b_m(i), b_n(j)) > k$. Indeed, this implies that (A) can not hold for i, j, or equivalently, (B) holds.

<u>Case 1</u>: If $i < j$.

Proof of case. Note that $b_0(i) < b_1(i) < b_2(i) < b_2(j)$. Then,

$$k < \rho(b_0(i), b_1(i)) \leqslant \max(\,\rho(b_0(i), b_2(j)),\ \rho(b_1(i), b_2(j))\,).$$

In other words, either $\rho(b_0(i), b_2(j)) > k$ or $\rho(b_1(i), b_2(j)) > k$. □

<u>Case 2</u>: If $i > j$ and $b_2(j) < b_0(i)$.

Proof of case. Here we also have that $b_1(j) < b_2(j)$. Hence,

$$k < \rho(b_1(j), b_2(j)) \leqslant \max(\,\rho(b_0(i), b_1(j)),\ \rho(b_0(i), b_2(j))\,).$$

As in the previous case, since either $\rho(b_0(i), b_1(j)) > k$ or $\rho(b_0(i), b_1(j)) > k$ then B must hold. □

<u>Case 3</u>: If $i > j$ and $b_0(i) < b_2(j)$.

Proof of case. Here, as $b_0(i) < b_1(i)$ the inequality associated to the first case also holds. The rest of the argument is equal to the first two cases. □

■

Definition 2.1.16. Let $\rho : X^2 \longrightarrow \omega$ be an ordinal metric and let $F, G \in \mathrm{FIN}(X)$. We say that F and G are *ρ-isomorphic* if $|F| = |G|$ and for all $i, j < |F|$,

$$\rho(F(i), F(j)) = \rho(G(i), G(j)).$$

Equivalently, if $h : F \longrightarrow G$ is the only increasing bijection[3] then

$$\rho(\alpha, \beta) = \rho(h(\alpha), h(\beta))$$

for all $\alpha, \beta \in F$.

Definition 2.1.17 (Homogeneous metric)**.** Let $\rho : X^2 \longrightarrow \omega$ be an ordinal metric. We say that ρ is *homogeneous* if all $F, G \in \mathrm{FIN}(X)$ maximally closed sets with the same diameter are ρ-isomorphic.

Lemma 2.1.18. *Suppose that $\rho : X^2 \longrightarrow \omega$ is a locally finite and homogeneous ordinal metric. If $F, G \in FIN(X)$ are two closed sets (not necessarily maximal) with $|F| = |G|$ and $\rho^F = \rho^G$ then F and G are ρ-isomorphic.*

---

[3]Note that if $|F| = |G|$ then h is defined by the formula $h(F(i)) = G(i)$.

Proof. Let $k = \rho^F = \rho^G$. As ρ is locally finite, there are maximally closed sets $\overline{F}, \overline{G} \in$ FIN(X) end-extending F and G respectively and such that $\rho^{\overline{F}} = \rho^{\overline{G}} = k$. In particular, this means that $F(i) = \overline{F}(i)$ and $G(i) = \overline{G}(i)$ for each $i < |F|$. Now, since ρ is homogeneous then $\rho(F(i), F(j)) = \rho(\overline{F}(i), \overline{F}(j)) = \rho(\overline{F}(i), \overline{F}(j)) = \rho(G(i), G(j))$ for any $i, j < |F| \leqslant |\overline{F}|$. This finishes the proof. ∎

Remark 2.1.19. If ρ, F and G are as in the previous lemma and $\phi : F \longrightarrow G$ is the only increasing bijection then $\rho^H = \rho^{\phi[H]}$ for each $H \subseteq F$. As F and G are closed it is easy to see that $\phi[H]$ is closed whenever H is. This is because $\rho^H \leqslant \rho^F$. Furthermore, if H is maximally closed then $\phi[H]$ is also maximally closed. This is because $\phi[H]$ can be extended to a maximally closed set $\overline{\phi[H]}$ of diameter ρ^H. By homogeneity, such set has the same cardinality as H. But then $\phi[H] = \overline{\phi[H]}$.

Proposition 2.1.20. *Let $\rho : X^2 \longrightarrow \omega$ be an ordinal metric and let $\mathcal{A} \subseteq$ FIN(X) be a family of closed sets of diameter k. If $F \in$ FIN(X) is such that $\rho^F \leqslant k$ and $F \subseteq \bigcup \mathcal{A}$ then there is $G \in \mathcal{A}$ for which $F \subseteq G$.*

Proof. By hypothesis, there is $G \in \mathcal{A}$ with $\max(F) \in G$. As $\rho^F \leqslant k = \rho^G$ and G is closed, we know that $F \cap G \sqsubseteq F$ due to the point (6) of Proposition 2.1.7. It follows directly that $F \subseteq G$. ∎

As a corollary of the previous proposition we get that there are no leftovers in the unions of maximally closed sets of the same diameter.

Corollary 2.1.21. *Let $\rho : X^2 \longrightarrow \omega$ be an ordinal metric and let $F \in$ FIN(X). If $i < \rho^F$ and $\mathcal{A} \subseteq$ FIN(X) is a family of maximally closed sets of diameter i such that $F = \bigcup \mathcal{A}$, then*

$$\mathcal{A} = \{ G \in \text{FIN}(X) : G \subseteq F, G \text{ is maximally closed and } \rho^G = i \}.$$

Proof. We only need to prove the inclusion from right to left. For this, let $G \in$ FIN(X) be such that $G \subseteq F$, G is maximally closed and $\rho^G = i$. By Proposition 2.1.20 we know there is $H \in \mathcal{A}$ with $G \subseteq H$. As G is maximally closed and $\rho^G = i = \rho^H$, then $G = H$. This finishes the proof. ∎

Definition 2.1.22 (Regular metric). Let $\rho : X^2 \longrightarrow \omega$ be an ordinal metric. We say that ρ is *regular* if for each $k \in \omega \backslash 1$ and each maximally closed set $F \in$ FIN(X) of diameter k there are $j_F \in \omega \backslash 2$ and $F_0, \ldots, F_{j_F - 1} \in$ FIN(X) such that:

- $F = \bigcup\limits_{i < j_F} F_i$,

- For each $i < j_F$, F_i is a maximal closed set with $\rho^{F_i} = k - 1$,

- $\langle F_i \rangle_{i < j_F}$ forms a root-tail-tail Δ-system with root $R(F)$. That is,

$$R(F) < F_0 \backslash R(F) < \ldots < F_{j_F - 1} \backslash R(F).$$

Remark 2.1.23. Observe that both the number j_F and sequence $\langle F_i \rangle_{i < j_F}$ are unique for each F due to Corollary 2.1.21. So from now on, we will call j_F the decomposition number of F and $\langle F_i \rangle_{i < j_F}$ its canonical decomposition.

Lemma 2.1.24. *Let $\rho : X^2 \longrightarrow \omega$ be a regular, locally finite ordinal metric. If $F \in FIN(X)$ is a closed set with positive diameter and $i < \rho^F$, then there is a maximally closed set $G \in FIN(X)$ with $\rho^G = i$ and $G \sqsubseteq F$.*

Proof. The proof is carried by induction over $k = \rho^F$. For this, fix $k \in \omega$ and suppose that we have proved the lemma for each closed set $H \in \mathrm{FIN}(X)$ with $\rho^H < k$. Let $F \in \mathrm{FIN}(X)$ be a closed set of diameter k and let $i < \rho^F$. Furthermore, $R(F)$ is also uniquely determined so we will call it the root of F.

As ρ is locally finite, there is a maximally closed set $\overline{F}$ with $F \sqsubseteq \overline{F}$ and $\rho^{\overline{F}} = k$. Since ρ is regular and $k > 0$ we can consider $j_{\overline{F}} \in \omega \backslash 2$ and $\overline{F}_0, \ldots, \overline{F}_{j_{\overline{F}}-1} \in \mathrm{FIN}(X)$ as in Definition 2.1.22. Now, $F \not\subseteq \overline{F}_0$ as $\rho^{\overline{F}_0} = k - 1$. As both F and $\overline{F}_0$ are initial segments of $\overline{F}$ it follows that $\overline{F}_0 \sqsubseteq F$. Observe that if $i = k - 1$ then $\overline{F}_0$ testifies the conclusion of the Lemma. Otherwise, if $i < k - 1$ we can use the induction hypotheses to get a maximally closed set G for which $\rho^G = i$ and $G \sqsubseteq \overline{F}_0 \sqsubseteq F$. This finishes the proof. ∎

In this way, for a regular, locally finite ordinal metric, we can not "skip a distance" in a closed set.

Corollary 2.1.25. *Let $\rho : X^2 \longrightarrow \omega$ be a regular, locally finite ordinal metric. If $F \in FIN(X)$ is closed and $i \leqslant \rho^F$, then there are $\alpha, \beta \in F$ for which $\rho(\alpha, \beta) = i$.*

Theorem 2.1.26. *Let $\rho : X^2 \longrightarrow \omega$ be a regular, locally finite and homogeneous ordinal metric. If $F, G \in FIN(X)$ are closed sets of the same cardinality then $\rho^F = \rho^G$.*

Proof. Suppose towards a contradiction that $\rho^F \neq \rho^G$. Without loss of generality we can assume that $\rho^F < \rho^G$. As ρ is locally finite there is maximally closed set $\overline{F}$ with $F \sqsubseteq \overline{F}$ and $\rho^{\overline{F}} = \rho^F$. Due to Lemma 2.1.24 there is also a maximally closed set H such that $H \sqsubseteq G$ and $\rho^H = \rho^F$. Observe that $|H| < |G|$. On the other hand, by the homogeneity of ρ we have that $|\overline{F}| = |H|$. Thus, $|F| \leqslant |\overline{F}| = |H| < |G|$ which is a contradiction. This finishes the proof. ∎

Corollary 2.1.27. *Let $\rho : X^2 \longrightarrow \omega$ be a regular, locally finite and homogeneous ordinal metric. If $\alpha, \beta \in X$ and $k \in \omega$ is such that $\|\alpha\|_k = \|\beta\|_k$ then $(\alpha)_k$ and $(\beta)_k$ are ρ-isomorphic.*

Proof. By Theorem 2.1.26 it follows that $\rho^{(\alpha)_k} = \rho^{(\beta)_k}$ so by Lemma 2.1.18 we are done. ∎

2.2   Construction schemes

By means of Lemma 2.1.18 and Theorem 2.1.26, closed sets in regular locally finite homogeneous ordinal metrics are completely determined by their size. Given such a metric ρ, let us call m_k^ρ the common cardinality of maximally closed sets of diameter k. The sequence $\langle m_k^\rho \rangle_{k \in \omega}$ is strictly increasing and the first point of such sequence, that is m_0^ρ, is 1 due the point (a) of Definition 2.1.4. A closed set will have diameter $k + 1$ if and only if its cardinality lies in the interval $(m_k^\rho, m_{k+1}^\rho]$, and will be maximal only when its size is exactly m_{k+1}^ρ.
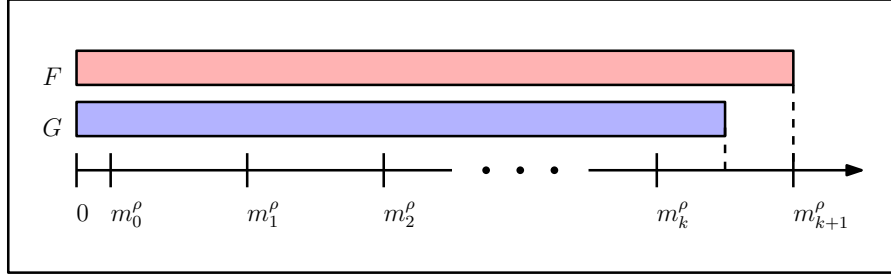
*Here, F represents a maximally closed set of diameter $k + 1$, while G
represents a closed (not maximal) set of diameter $k + 1$.*

By homogeneity we also know that for each maximally closed set F of positive
diameter k, both the decomposition number j_F as well as the cardinality of the root
$|R(F)|$ are uniquely determined by k. Hence, we can rename these two numbers as n_k^ρ
and r_k^ρ respectively.
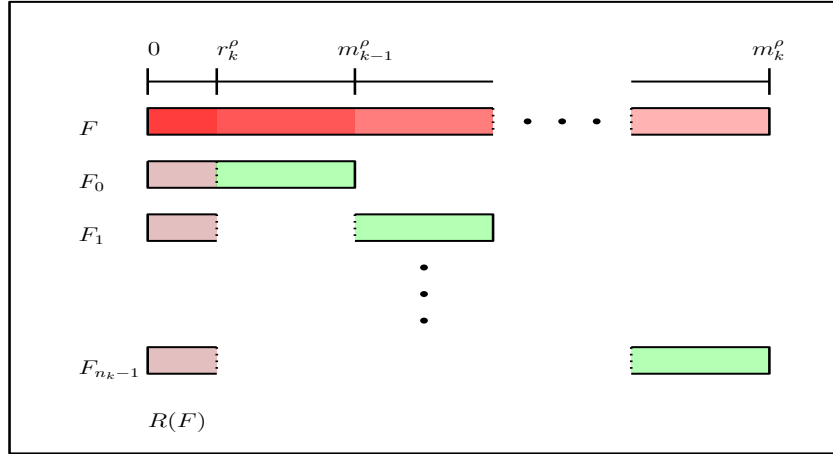
*Here, F represents a maximally closed of diameter k. Each piece of its
canonical decomposition is painted into two colors. The color brown
represents the root of F and the green one represents the remaining part.*

From the previous picture it is clear that the numbers m_k^ρ, n_k^ρ and r_k^ρ are related through
the equation:

$$m_{k+1}^\rho = r_{k+1}^\rho + (m_k^\rho - r_{k+1}^\rho)n_{k+1}^\rho.$$

An abstraction of the properties satisfied by the sequence $\langle m_k^\rho, n_{k+1}^\rho, r_{k+1}^\rho \rangle_{k\in\omega}$ lead to
the next definition.

Definition 2.2.1 (Type)**.** We call a sequence $\tau = \langle m_k, n_{k+1}, r_{k+1} \rangle_{k\in\omega} \subseteq \omega^3$ a *type* if:

(a) $m_0 = 1$,

(b) $\forall k \in \omega \backslash 1 \Big(n_k \geqslant 2 \Big)$,

(c) $\forall k \in \omega \Big(m_k > r_{k+1} \Big)$,

(d) $\forall k \in \omega \Big(m_{k+1} = r_{k+1} + (m_k - r_{k+1})n_{k+1} \Big)$.

We say that τ is *good* if:

(e) $\forall r \in \omega \ \exists^\infty k \in \omega \big(r_k = r \big)$,

Additionally, we say that a partition of ω, namely $\mathcal{P}$, is compatible with τ if:

(e') $\forall P \in \mathcal{P} \ \forall r \in \omega \ \exists^\infty k \in P(r_k = r)$.

Now, let $\mathcal{F}^\rho$ be the collection of all maximally closed sets and consider it as ordered set with respect to $\subseteq$. First observe that $\mathcal{F}^\rho$ is cofinal in $\mathrm{FIN}(X)$ by Remark 2.1.11. Note also that $\mathcal{F}^\rho$ is well-founded since it consists of finite sets. Using Lemma 2.1.24 one can show recursively that $rank(F) = \rho^F$ for each $F \in \mathcal{F}^\rho$.

We now define a construction scheme as a family of finite sets of X satisfying the same basic properties as $\mathcal{F}^\rho$.[4] But first we make a brief comment regarding the notation. Each family $\mathcal{F} \subseteq \mathrm{FIN}(X)$ is well-founded with respect to $\subseteq$ (moreover, $Ht(\mathcal{F}) \leqslant \omega$). In this way, we can define $\mathcal{F}_k$ as in the preliminaries for each $k \in \omega$. That is, $\mathcal{F}_k = \{F \in \mathcal{F} : rank(F) = k\}$.

Definition 2.2.2 (Construction scheme). Let $\tau = \langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$ be a type, and X be a set of ordinals. We say that $\mathcal{F} \subseteq \mathrm{FIN}(X)$ is a *construction scheme* (or simply *a scheme*) over X of type τ if:

(a) $\mathcal{F}$ is cofinal in $\mathrm{FIN}(X)$,

(b) $\forall k \in \omega \ \forall F \in \mathcal{F}_k \big(|F| = m_k \big)$,

(c) $\forall k \in \omega \ \forall F, E \in \mathcal{F}_k \big(E \cap F \sqsubseteq E, F \big)$,

(d) $\forall k \in \omega \ \forall F \in \mathcal{F}_{k+1} \ \exists F_0, \dots, F_{n_{k+1}-1} \in \mathcal{F}_k$ such that

$$F = \bigcup_{i < n_{k+1}} F_i.$$

Moreover, $\langle F_i \rangle_{i < n_{k+1}}$ forms a Δ-system with root $R(F)$ such that $|R(F)| = r_{k+1}$ and

$$R(F) < F_0 \backslash R(F) < \dots < F_{n_{k+1}-1} \backslash R(F).$$

If $n \in \omega$ and $n_{k+1} = n$ for each $k \in \omega$, we will call $\mathcal{F}$ an *n-construction scheme* (or simply an *n-scheme*).[5]

Remark 2.2.3. If $F \in \mathcal{F}_{k+1}$ and $\langle F_i \rangle_{i < n_{k+1}}$ is as in the point (d) of the previous definition, then for each $i < n_{k+1}$,

$$F_i = F[r_{k+1}] \cup F\big[[a_i, a_{i+1}) \big]$$

where $a_i = r_{k+1} + i \cdot (m_k - r_{k+1})$. In particular this means that the sequence $\langle F_i \rangle_{i < n_{k+1}}$ is uniquely determined. For this reason, we will call it *the canonical decomposition of F*.

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[4]This approach is not historically accurate. In [112], construction schemes were defined in the second section, applied in the next three sections, and finally related to ordinal metrics in the sixth one.

[5]2-construction schemes are the objects which we also know as $(\omega, 1)$-gap morasses.

By the previous discussion and results in the first section of this chapter we conclude:

Proposition 2.2.4. *Let $\rho : X^2 \longrightarrow \omega$ be a regular, locally finite and homogeneous ordinal metric. The set*

$$\mathcal{F}^\rho = \{ F \in FIN(X) : F \text{ is maximally closed} \}$$

is a construction scheme over X of type $\langle m_k^\rho, n_{k+1}^\rho, r_{k+1}^\rho \rangle_{k \in \omega}$.

The following Theorem was proved in [112]. Another proof can be found in [69]. We will provide a complete proof of it in the Subsection 5.1.1 for the convenience of the reader.

Theorem 2.2.5. *For any good type there is a construction scheme over ω_1 of that type.*

Our next goal is to prove the converse of Proposition 2.2.4. That is, we will prove that every construction scheme naturally defines a regular, locally finite and homogeneous ordinal metric. Moreover, we will show the construction scheme coincides with the family of maximally closed sets of such metric. In order to do that, we will need some previous results.

The next lemma is an easy consequence of the condition (d) in Definition 2.2.2.

Lemma 2.2.6. *Let $\mathcal{F}$ be a construction scheme over X, $l \in \omega$ and $F \in \mathcal{F}_l$. For each $k \leqslant l$, it happens that $F = \bigcup \{ H \in \mathcal{F}_k : H \subseteq F \}$.*

There is a version of condition (c) in Definition 2.2.2 in the case where the sets have different rank.

Lemma 2.2.7. *Let $\mathcal{F}$ be a construcion scheme over X and let $k \leqslant l \in \omega$. If $G \in \mathcal{F}_k$ and $F \in \mathcal{F}_l$ then $G \cap F \sqsubseteq G$.*

Proof. As a consequence of Lemma 2.2.6 we know that

$$G \cap F = G \cap \left(\bigcup \{ H \in \mathcal{F}_k : H \subseteq F \} \right)$$
$$= \bigcup_{H \in \mathcal{F}_k \cap \mathscr{P}(F)} G \cap H.$$

As each of the sets forming the union of the last equality is an initial segment of G by condition (c) of Definition 2.2.2, such union is also an initial segment of G. ∎

Remark 2.2.8. In general it is not true that if $k < l$, $G \in \mathcal{F}_k$ and $F \in \mathcal{F}_l$ then $G \cap F$ is an initial segment of F. An example of this ocurrs when G is the second piece of the canonical decomposition of F, namely F_1.

Proposition 2.2.9. *Let $\mathcal{F}$ be a construction scheme over X and let $k \in \omega$ be such that $m_k \leqslant |X|$. Then $X = \bigcup \mathcal{F}_k$.*

Proof. Let $\alpha \in X$ and let Y be a subset of X of cardinality m_k. Due to condition (a) of Definition 2.2.2 there is $l \in \omega$ and $F \in \mathcal{F}_l$ for which $\{\alpha\} \cup Y \subseteq F$. Note that $l \geqslant k$ as $|Y| = m_k$. Therefore, by Lemma 2.2.6 we conclude that $F = \bigcup\{G \in \mathcal{F}_k \, : \, G \subseteq F\}$. In particular $\alpha \in G$ for some $G \in \mathcal{F}_k$ with $G \subseteq F$. This finishes the proof. ∎

Remark 2.2.10. By applying the previous proposition to the case where $k = 0$ we conclude that $[X]^1 = \mathcal{F}_0$.

Definition 2.2.11 (The ρ-function)**.** Let $\mathcal{F}$ be a construction scheme over X. We define $\rho_{\mathcal{F}} : X^2 \longrightarrow \omega$ as:

$$\rho_{\mathcal{F}}(\alpha, \beta) = \min(\, k \in \omega \, : \, \exists F \in \mathcal{F}_k(\, \{\alpha, \beta\} \subseteq F \,)\,).$$

If $\mathcal{F}$ is clear from context, we will write $\rho_{\mathcal{F}}$ simply as ρ.

Remark 2.2.12. Note that $\rho(\alpha, \beta)$ is well defined since $\mathcal{F}$ is cofinal in $\mathrm{FIN}(X)$.

Lemma 2.2.13. *Let $\mathcal{F}$ be a construction scheme over X. If $\beta \in X$ and $k \in \omega$, then*

$$(\beta)_k = F \cap (\beta + 1)$$

for each $F \in \mathcal{F}_k$ with $\beta \in F$.

Proof. By Definition 2.2.11 it is clear that $F \cap (\beta + 1) \subseteq (\beta)_k$. It only remains to prove the other inclusion. For this, let $\alpha \in (\beta)_k$ and $G \in \mathcal{F}_{\rho(\alpha, \beta)}$ be such that $\{\alpha, \beta\} \subseteq G$. By the definition of the k-closure we know that $\rho(\alpha, \beta) \leqslant k$. Hence, $G \cap F \sqsubseteq G$ due to Lemma 2.2.7. As $\beta \in G \cap F$ and $\alpha \leqslant \beta$ it must happen that $\alpha \in G \cap F$. In particular $\alpha \in F$, so we are done. ∎

Proposition 2.2.14. *Let $\mathcal{F}$ be a construction scheme over X. Then $\rho = \rho_{\mathcal{F}}$ is an ordinal metric.*

Proof. The condition (b) of Definition 2.1.4 is trivially satisfied, the condition (a) is a consequence of the Remark 2.2.10 and the condition (d) follows from Lemma 2.2.13. Thus, we only need to prove that the condition (c) of such definition also holds. For this, let $\alpha, \beta, \gamma \in X$ be such that $\alpha \leqslant \beta, \gamma$. Consider $F \in \mathcal{F}_{\rho(\alpha, \gamma)}$ and $G \in \mathcal{F}_{\rho(\beta, \gamma)}$ for which $\{\alpha, \gamma\} \subseteq F$ and $\{\beta, \gamma\} \subseteq G$. We need to consider two cases.

<u>Case 1</u>: If $\rho(\alpha, \gamma) \leqslant \rho(\beta, \gamma)$.

Proof of case. Here $F \cap G \sqsubseteq F$ by Lemma 2.2.7. As $\gamma \in F \cap G$ and $\alpha \leqslant \gamma$ we have that $\alpha \in F \cap G$. In particular $\alpha \in G$ so $\{\alpha, \beta\} \subseteq G$. Therefore $\rho(\alpha, \beta) \leqslant \rho(\beta, \gamma)$. □

<u>Case 2</u>: If $\rho(\beta, \gamma) \leqslant \rho(\alpha, \gamma)$.

Proof of case. In this case $F \cap G \sqsubseteq G$. If $\beta \leqslant \gamma$ then $\beta \in F \cap G$ because $\gamma \in F \cap G$. Therefore $\{\alpha, \beta\} \subseteq G$ which means $\rho(\alpha, \beta) \leqslant \rho(\beta, \gamma)$. On the other hand, if $\beta \geqslant \gamma$ we can use Proposition 2.2.9 to pick $F' \in \mathcal{F}_{\rho(\alpha, \gamma)}$ for which $\beta \in F'$. Since $F' \cap (\beta + 1) = (\beta)_{\rho(\alpha, \gamma)}$ by Lemma 2.2.13 then $\gamma \in F'$. Again, by the same lemma, $F' \cap (\gamma + 1) = (\gamma)_{\rho(\alpha, \gamma)}$ which means that $\alpha \in F'$. Consequently $\{\alpha, \beta\} \subseteq F'$ so $\rho(\alpha, \beta) \leqslant \rho(\alpha, \gamma)$. □

■

Proposition 2.2.15. *Let $\mathcal{F}$ be a construction scheme over X, $\rho = \rho_{\mathcal{F}}$ and let $F \in \mathcal{F}_k$ for some $k > 0$. Consider $\langle F_i \rangle_{i < n_k}$ the canonical decomposition of F as described in point (d) of Definition 2.2.2. If $\alpha \in F_i \backslash R(F)$ and $\beta \in F_j \backslash R(F)$ for $i < j < n_k$ then $\rho(\alpha, \beta) = k$.*

Proof. First observe that $\rho(\alpha, \beta) \leqslant k$ as testified by F. Now, $(\beta)_{k-1} = F_j \cap (\beta + 1)$ due to the Lemma 2.2.13. In this way, $\alpha \notin (\beta)_{k-1}$, which means that $\rho(\alpha, \beta) > k - 1$. This finishes the proof. ■

Lemma 2.2.16. *Let $\mathcal{F}$ be a construction scheme over X and $\rho = \rho_{\mathcal{F}}$. For every $k \in \omega$ and $F \in \mathcal{F}_k$, we have that $\rho^F = k$.*

Proof. For $k = 0$ the result follows from Remark 2.2.10, so we will only consider the case where $k > 0$. Let $F \in \mathcal{F}_k$. By Definition 2.2.11 it should be clear that $\rho^F \leqslant k$. To prove the other inequality let $\langle F_i \rangle_{i < n_k}$ be the canonical decomposition of F as described in point (d) of Definition 2.2.2. Now, consider $\alpha \in F_0 \backslash R(F)$ and $\beta \in F_1 \backslash R(F)$. By Proposition 2.2.15, $\rho(\alpha, \beta) = k$. Hence $k \leqslant \rho(\alpha, \beta) \leqslant \rho^F$. ■

Lemma 2.2.17. *Let $\mathcal{F}$ be a construction scheme over X and $\rho = \rho_{\mathcal{F}}$. Each element of $\mathcal{F}$ is maximally closed with respect to ρ.*

Proof. Let $k \in \omega$ and $F \in \mathcal{F}_k$. Consider $\alpha = \max(F)$. By Lemma 2.2.13, $F = F \cap (\alpha + 1) = (\alpha)_k$. In this way F is closed thanks to the point (4) of Proposition 2.1.7. To see that F is maximally closed let $G \in \mathrm{FIN}(X)$ be a closed set with $\rho^G = k$ and $F \subseteq G$. Consider $\beta = \max(G)$. Through the use of Proposition 2.2.9 we can take $H \in \mathcal{F}_k$ for which $\beta \in H$. Then $(\beta)_k = H \cap (\beta + 1)$ due to Lemma 2.2.13. Furthermore, by the point (1) of Proposition 2.1.7 we know that $\rho(\delta, \beta) \leqslant k$ for each $\delta \in G$. In particular this means that $F \subseteq G \subseteq H$. Since both F and H have size m_k, we conclude that $F = H = G$. This finishes the proof. ■

Proposition 2.2.18. *Let $\mathcal{F}$ be a construction scheme over X and $\rho = \rho_{\mathcal{F}}$. Then*

$$\mathcal{F}_k = \{\, H \in \mathrm{FIN}(X) : H \text{ is maximally closed and } \rho^H = k \,\}$$

for each $k \in \omega$.

Proof. The inclusion from left to right is a direct consequence of Lemmas 2.2.16 and 2.2.17. To prove the one from right to left, let $H \in \mathrm{FIN}(X)$ be a maximally closed set of diameter k. Consider $\alpha = \max(H)$ and take $F \in \mathcal{F}_k$ for which $\alpha \in F$. Then $(\alpha)_k = F \cap (\alpha + 1) = F$. This means that $H \subseteq F$ due to the point (1) of Proposition 2.1.7. Since $\rho^F = k$ by Lemma 2.2.16 and H is maximally closed, it follows that $H = F$. Thus, the proof is over. ■

Theorem 2.2.19. *Let $\mathcal{F}$ be a construction scheme over X. Then $\rho = \rho_{\mathcal{F}}$ is locally finite, regular and homogeneous.*

Proof. The function ρ is locally finite because each closed set H is contained in $F \cap (\max(H)+1)$ where $F \in \mathcal{F}_{\rho^H}$ is such that $\max(H) \in F$. In particular, this means that $|H|$ is bounded by m_{ρ^H}. Now, ρ is regular due to Proposition 2.2.18 and the point (d) of Definition 2.2.2. The only thing left to do is to prove that ρ is homogeneous. This proof is carried by induction over the diameter k of the maximally closed sets involved. The case where $k = 0$ is trivial as maximally closed sets of diameter 0 are singletons. So suppose that $k \in \omega$, and we have proved homogeneity for all maximally closed sets of diameter less than k. Let F, G be two maximally closed sets with $\rho^F = \rho^G = k$. By Proposition 2.2.18, $F, G \in \mathcal{F}_k$. Consequently $|F| = |G| = m_k$. Thus, we can let $h : F \longrightarrow G$ be the only increasing bijection. We claim that $\rho(\alpha, \beta) = \rho(h(\alpha), h(\beta))$ for each $\alpha, \beta \in F$. For this, take $\langle F_i \rangle_{i < n_k}$ and $\langle G_i \rangle_{i < n_k}$ the canonical decompositions of F and G respectively. It is easy to see that $h[F_i] = G_i$ for each $i < n_k$ and that $h[R(F)] = R(G)$. Consequently, if $\alpha, \beta \in F_i$ for some $i < n_k$ we can use the induction hypothesis over F_i and G_i to conclude that $\rho(\alpha, \beta) = \rho(h(\alpha), h(\beta))$. On the other hand, if there is no i for which both α and β belong to F_i, there are distinct $i, j < n_k$ for which $\alpha \in F_i \backslash R(F)$ and $\beta \in F_j \backslash R(F)$. Observe that $h(\alpha) \in G_i \backslash R(G)$ and $h(\beta) \in G_j \backslash R(F)$. Therefore, by Proposition 2.2.15 we have that $\rho(\alpha, \beta) = k = \rho(h(\alpha), h(\beta))$. This finishes the proof. ∎

The following corollary follows from the proof of the previous theorem as well as of Proposition 2.2.4.

Corollary 2.2.20. *If $\mathcal{F}$ is a construction scheme over X and $\rho : X^2 \longrightarrow \omega$ is a locally finite, homogeneous and regular ordinal metric, then:*

- $\mathcal{F}^{\rho_{\mathcal{F}}} = \mathcal{F}$,

- $\rho_{\mathcal{F}^\rho} = \rho$.

2.3   Two more canonical functions

Apart from the ordinal metric associated to a construction scheme, there are two important functions that we need to analyze before we enter the world of applications. The first one being the Δ function and the second one being the Ξ function.

For the rest of this section, fix a construction scheme $\mathcal{F}$ over X and let $\rho = \rho_{\mathcal{F}}$ be its associated ordinal metric.

The Δ function just measures the least moment in which the k-cardinality of two ordinals differs.

Definition 2.3.1 (The Δ function). We define $\Delta : X^2 \longrightarrow \omega + 1$ as:

$$\Delta(\alpha, \beta) = \begin{cases} \min(\, k \in \omega \,:\, \|\alpha\|_k \neq \|\beta\|_k \,) & \text{if } \alpha \neq \beta \\ \omega & \text{if } \alpha = \beta \end{cases}$$

Remark 2.3.2. Δ is well defined since $\|\alpha\|_{\rho(\alpha,\beta)} \neq \|\beta\|_{\rho(\alpha,\beta)}$ whenever $\alpha \neq \beta$. Moreover, $\Delta(\alpha, \beta) \leqslant \rho(\alpha, \beta)$.

Lemma 2.3.3. *Let $\alpha, \beta \in X$ be distinct ordinals and $k \in \omega$. If $\|\alpha\|_k = \|\beta\|_k$ then $k < \Delta(\alpha, \beta)$.*

Proof. By Corollary 2.1.27 $(\alpha)_k$ is ρ-isomorphic to $(\beta)_k$. Let $h : (\alpha)_k \longrightarrow (\beta)_k$ be the only increasing bijection. Note that $h(\alpha) = \beta$. Now consider an arbitrary $i \leqslant k$. Then $(\alpha)_i \subseteq (\alpha)_k$. In this way, $h[(\alpha)_i] = (h(\alpha))_i = (\beta)_i$. This means that $\|\alpha\|_i = \|\beta\|_i$. Therefore $k < \Delta(\alpha, \beta)$. ∎

Lemma 2.3.4. *Let $\alpha, \beta, \delta \in X$ be distinct ordinals such that $\Delta(\alpha, \beta) < \Delta(\beta, \delta)$. Then $\Delta(\alpha, \delta) = \Delta(\alpha, \beta)$.*

Proof. $\Delta(\alpha, \delta) \leqslant \Delta(\alpha, \beta)$ since $\|\alpha\|_{\Delta(\alpha,\beta)} \neq \|\beta\|_{\Delta(\alpha,\beta)} = \|\delta\|_{\Delta(\alpha,\beta)}$. Now, to see that $\Delta(\alpha, \delta) \geqslant \Delta(\alpha, \beta)$ take an arbitrary $i < \Delta(\alpha, \beta)$. Then $\|\alpha\|_i = \|\beta\|_i = \|\delta\|_i$. This finishes the proof. ∎

Lemma 2.3.5. *Let $\alpha, \beta \in X$ be distinct ordinals, $k < \Delta(\alpha, \beta)$ and $h : (\alpha)_k \longrightarrow (\beta)_k$ be the only increasing bijection.*

| *If $\delta \in (\alpha)_k$ then:*	*In other words, if $i < \|\alpha\|_k$ then:*
(a) $\Delta(\delta, h(\delta)) \geqslant \Delta(\alpha, \beta)$,	*(a) $\Delta((\alpha)_k(i), (\beta)_k(i)) \geqslant \Delta(\alpha, \beta)$,*
(b) $\rho(\alpha, \beta) \geqslant \rho(\delta, h(\delta))$.	*(b) $\rho(\alpha, \beta) \geqslant \rho((\alpha)_k(i), (\beta)_k(i))$.*

Proof. Proof of (a). Let $l = \Delta(\alpha, \beta) - 1$ and $h' : (\alpha)_l \longrightarrow (\beta)_l$ be the only increasing bijection. As $k < \Delta(\alpha, \beta)$ then $(\alpha)_k \subseteq (\alpha)_l$ and $h'[(\alpha)_k] = (\beta)_k$. Hence $h'|_{(\alpha)_k} = h$ so $h'(\delta) = h(\delta)$. Therefore $h'[(\delta)_l] = (h(\delta))_l$. In this way, $\|\delta\|_l = \|h(\delta)\|_l$. Thus, by Lemma 2.3.3 we conclude that $\Delta(\alpha, \beta) \leqslant \Delta(\delta, \phi(\delta))$. □

Proof of (b). Let $F \in \mathcal{F}_{\rho(\alpha,\beta)}$ be such that $\alpha, \beta \in F$. Since $k < \Delta(\alpha, \beta) \leqslant \rho(\alpha, \beta)$ then $(\alpha)_k \cup (\beta)_k \subseteq F$. In particular $\delta, h(\delta) \in F$ so $\rho(\alpha, \beta) \geqslant \rho(\delta, h(\delta))$. □

∎

Remark 2.3.6. Recall that if α and β are so that $\|\alpha\|_k = \|\beta\|_k$ and $h : (\alpha)_k \longrightarrow (\beta)_k$ is the only increasing bijection, then the following happens:

- $h(\alpha) = h(\beta)$,

- $(\alpha)_k \cap (\beta)_k \sqsubseteq (\alpha)_k, (\beta)_k,$

- $h|_{(\alpha)_k \cap (\beta)_k}$ is the identity function,

- $h[(\alpha)_k \backslash (\beta)_k] = (\beta)_k \backslash (\alpha)_k.$
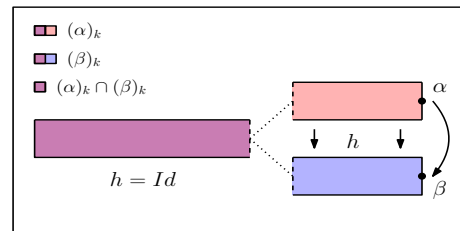
Lemma 2.3.7. *Let $\alpha, \beta \in X$ be distinct ordinals and $k < \Delta(\alpha, \beta)$. If $h : (\alpha)_k \longrightarrow (\beta)_k$ is the only increasing bijection and $\delta, \gamma \in (\alpha)_k$ are such that $\delta \leqslant \gamma$ and $h(\delta) \neq \delta$ then the following happens:*

(a) $h(\gamma) \neq \gamma$,

(b) $\rho(\alpha,\beta) \geqslant \rho(\gamma,h(\gamma)) \geqslant \rho(\delta,h(\delta)) \geqslant \Delta(\delta,h(\delta)) \geqslant \Delta(\gamma,h(\gamma)) \geqslant \Delta(\alpha,\beta)$.

Proof. Remember that $C = (\alpha)_k \cap (\beta)_k$ is an initial segment of both $(\alpha)_k$ and $(\beta)_k$. In this way, it is easy to see that $h(\xi) = \xi$ if and only if $\xi \in C$. Note that since $\delta \notin C$ and $\gamma > \delta$, then $\gamma \notin C$. Therefore $h(\gamma) \neq \gamma$. This proves the point (a).

The inequalities from (b) follow directly from Lemma 2.3.5 and the fact that $h|_{(\gamma)_k}$ is the only increasing bijection from $(\gamma)_k$ to $(h(\gamma))_k$.

■

Definition 2.3.8 (The Ξ function). Let $\alpha \in X$. $\Xi_\alpha : \omega \longrightarrow \omega \cup \{-1\}$ is the function defined as:

$$\Xi_\alpha(k) = \begin{cases} 0 & \text{if } k = 0 \\ -1 & \text{if } k > 0 \text{ and } \|\alpha\|_k \leqslant r_k \\ \frac{\|\alpha\|_k - \|\alpha\|_{k-1}}{m_{k-1} - r_k} & \text{otherwise} \end{cases}$$

It is not hard to check that if $k \in \omega \backslash 1$ and $F \in \mathcal{F}_k$ is such that $\alpha \in F$, then:

$$\Xi_\alpha(k) = \begin{cases} -1 & \text{if } \alpha \in R(F) \\ i & \text{if } \alpha \in F_i \backslash R(F) \end{cases}$$

The main reason for defining the Ξ function is that without appealing to any extra axioms it is really hard to give useful properties regarding the behavior of the k-cardinality of pairs of ordinals. The Ξ function reveals two important and natural critical points "leading the dance" of the k-cardinality of ordinals α and β as k grows larger. As the next lemma shows, below $\Delta(\alpha,\beta)$ and above $\rho(\alpha,\beta)$, this dance is smooth and pleasant whereas between $\Delta(\alpha,\beta)$ and $\rho(\alpha,\beta)$ it seems that almost anything is possible.
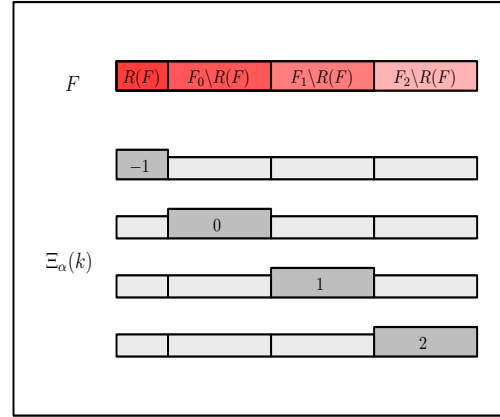
In here, F represents a member of $\mathcal{F}_k$ with $n_k = 3$.

Lemma 2.3.9. *Let $\alpha < \beta \in X$ and $k \in \omega \backslash 1$. Then:*

(a) *If $k < \Delta(\alpha,\beta)$, then $\Xi_\alpha(k) = \Xi_\beta(k)$.*

(b) *If $k = \rho(\alpha,\beta)$, then $0 \leqslant \Xi_\alpha(k) < \Xi_\beta(k)$.*

(c) *If $k > \rho(\alpha,\beta)$, then either $\Xi_\alpha(k) = -1$ or $\Xi_\alpha(k) = \Xi_\beta(k)$.*

(d) *If $k = \Delta(\alpha,\beta)$ then $0 \leqslant \Xi_\alpha(k) \neq \Xi_\beta(k) \geqslant 0$.*

Proof. (a) is direct from the definition of $\Delta(\alpha,\beta)$ so we will only prove the remaining points.

Proof of (b). Fix $F \in \mathcal{F}_{\rho(\alpha,\beta)}$ for which $\alpha, \beta \in F$. We know there are $i, j < n_k$ such that $\alpha \in F_i$ and $\beta \in F_j$. Since $F_i, F_j \in \mathcal{F}_{\rho(\alpha,\beta)-1}$, minimality of ρ implies that $\alpha \notin F_j$ and $\beta \notin F_i$. In particular, this means $i \neq j$ and $\alpha, \beta \notin R(F)$. Thus, by Definition 2.3.8 we have that $\Xi_\alpha(k) = i$ and $\Xi_\beta(k) = j$. Moreover, since $\alpha < \beta$ then $i < j$. $\square$

Proof of (c). For this, suppose that $\Xi_\alpha(k) \neq -1$ and let $F \in \mathcal{F}_k$ be such that $\alpha, \beta \in F$. Since $\alpha < \beta$, we also have $\Xi_\beta(k) \neq -1$. Thus, there is $j < n_k$ for which $\beta \in F_j \backslash R(F)$. Recall that $F_j \in \mathcal{F}_{k-1}$. In this way, $\alpha \in (\beta)_{k-1} = (\beta + 1) \cap F_j$. As $\alpha \notin R(F)$, this means $\Xi_\alpha(k) = j = \Xi_\beta(k)$. So we are done. $\square$

Proof of (d). Let $F, G \in \mathcal{F}_k$ be such that $\alpha \in F$ and $\beta \in G$. Also, let $h : F \longrightarrow G$ be the increasing bijection. Then $h[(\alpha)_i] = (h(\alpha))_i$ for any $i \leqslant k$. In this way,

$$k = \Delta(\alpha, \beta) = \Delta(h(\alpha), \beta) \leqslant \rho(h(\alpha), \beta) \leqslant k.$$

That is, $k = \rho(h(\alpha), \beta)$. So by the part (b) of this lemma we conclude that $\Xi_{h(\alpha)}(k)$ and $\Xi_\beta(k)$ are both distinct and greater or equal to 0. To finish just note that $\Xi_\alpha(k) = \Xi_{h(\alpha)}(k)$. $\square$

∎

As an easy consequence of the previous lemma we have the following corollary.

Corollary 2.3.10. *Suppose that $X = \omega_1$ and let $\alpha \in X$. Then:*

(1) $\Xi_\alpha(k) = -1$ *for infinitely many $k \in \omega$.*

(2) *If α is infinite, then $\Xi_\alpha(k) \geqslant 1$ for infinitely many $k \in \omega$.*

(3) *If τ (the type of $\mathcal{F}$) is a good type, then $\Xi_\alpha(k) \geqslant 0$ for infinitely many $k \in \omega$.*

Proof. In order to prove (3) just observe that since τ is a good type then $r_k = 0$ for infinitely many k's. For any such k it is necessarily true that $\Xi_\alpha(k) \geqslant 0$. We now prove the remaining points.

Proof of (1). Let $l \in \omega$. We will find $k > l$ for which $\Xi_\alpha(k) = -1$. Since $X = \omega_1$ there is an uncountable $S \subseteq \omega_1 \backslash \alpha$ and $s \in \omega$ such that $\rho(\alpha, \beta) = \rho(\alpha, \delta) = s$ for any $\beta, \delta \in S$. In virtue of Lemma 2.1.13 there are $\beta < \delta \in S$ for which $\rho(\beta, \delta) > \max(l, s)$. Let $k = \rho(\beta, \delta)$. By the point (b) in Lemma 2.3.9 we know that $0 \leqslant \Xi_\beta(k) < \Xi_\delta(k)$. If $\Xi_\alpha(k) \neq -1$ then we would have that $\Xi_\beta(k) = \Xi_\alpha(k) = \Xi_\delta(k)$ due to part (c) of that same lemma, which is impossible. Hence, we are done. $\square$

Proof of (2). Fix $l \in \omega$. We shall find $k > l$ for which $\Xi_\alpha(k) \geqslant 1$. Since $(\alpha)_l$ is finite and α is infinite there is $\xi \in \alpha \backslash (\alpha)_l$. By the definition of the l-closure we have that $k = \rho(\xi, \alpha) > l$. Then $\Xi_\alpha(k) \geqslant 1$ do to the point (b) of Lemma 2.3.9. $\square$

∎

Lemma 2.3.11. *Let $\alpha, \beta \in X$ be distinct ordinals, $k < \Delta(\alpha, \beta)$ and $h : (\alpha)_k \longrightarrow (\beta)_k$ be the only increasing bijection. If $\gamma \in (\alpha)_k$ is such that $\gamma \neq h(\gamma)$ then the following are equivalent:*

(a) $\Delta(\gamma, h(\gamma)) > \Delta(\alpha, \beta)$.

(b) $\Xi_\gamma(\Delta(\alpha, \beta)) = -1$.

Furthermore, if $\Xi_\gamma(\Delta(\alpha, \beta)) \geqslant 0$ then $\Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_\alpha(\Delta(\alpha, \beta))$ and $\Xi_{h(\gamma)}(\Delta(\alpha, \beta)) = \Xi_\beta(\Delta(\alpha, \beta))$.

Proof. *Proof of* $(a) \Rightarrow (b)$. We argue by contradiction. Suppose that $\Xi_\gamma(\Delta(\alpha, \beta)) \geqslant 0$. Since $\rho(\alpha, \gamma) \leqslant k < \Delta(\alpha, \beta)$ then $\Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_\alpha(\Delta(\alpha, \beta)) \geqslant 0$ by the part (c) of Lemma 2.3.9. Now, since we are assuming that $\Delta(\gamma, h(\gamma)) > \Delta(\alpha, \beta)$, then $0 \leqslant \Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_{h(\gamma)}(\Delta(\alpha, \beta))$ due to the point (a) of Lemma 2.3.9. Thus, we can argue in the same way as before to conclude that $\Xi_{h(\gamma)}(\Delta(\alpha, \beta)) = \Xi_\beta(\Delta(\alpha, \beta))$. Therefore, according the part (d) of the same lemma,

$$\Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_\alpha(\Delta(\alpha, \beta)) \neq \Xi_\beta(\Delta(\alpha, \beta)) = \Xi_{h(\gamma)}(\Delta(\alpha, \beta)).$$

We conclude using the part (a) of Lemma 2.3.9 that $\Delta(\gamma, h(\gamma)) = \Delta(\alpha, \beta)$. $\square$

Proof of $(b) \Rightarrow (a)$. Suppose that $\Xi_\gamma(\Delta(\alpha, \beta)) = -1$. Then the conclusion of the part (d) of Lemma 2.3.9 can not hold when applied to γ, $h(\gamma)$ and $\Delta(\alpha, \beta)$. In virtue of this, $\Delta(\alpha, \beta)$ must be distinct from $\Delta(\gamma, h(\gamma))$. $\square$

$\blacksquare$

Corollary 2.3.12. *Let $\xi < \alpha < \beta \in X$. If $\rho(\xi, \beta) < \rho(\alpha, \beta)$ then $\rho(\xi, \alpha) < \rho(\alpha, \beta)$.*

Proof. Since ρ is an ordinal metric, $\rho(\xi, \alpha) \leqslant \max(\rho(\xi, \beta), \rho(\alpha, \beta)) = \rho(\alpha, \beta)$. Suppose towards a contradiction that $\rho(\alpha, \beta) = \rho(\xi, \alpha)$. Then by the part (b) of Lemma 2.3.9 we have that

$$0 \leqslant \Xi_\xi(\rho(\alpha, \beta)) < \Xi_\alpha(\rho(\alpha, \beta)) < \Xi_\beta(\rho(\alpha, \beta)).$$

On the other hand, since $\rho(\alpha, \beta) > \rho(\beta, \xi)$ then $\Xi_\xi(\rho(\alpha, \beta)) = -1$ or $\Xi_\xi(\rho(\alpha, \beta)) = \Xi_\beta(\rho(\alpha, \beta))$ due to the part (c) of Lemma 2.3.9. Both cases contradict the previous inequality so we are done. $\blacksquare$

The definition of the Ξ function can be extended (with some restrictions) to arbitrary elements of $\mathrm{FIN}(X)$. More precisely, by virtue of Proposition 2.1.20 we know that:

Lemma 2.3.13. *Let $A \in FIN(X)$, $k > \rho^A$ and $F \in \mathcal{F}_k$ such that $A \subseteq F_i$. Then there is $i < n_k$ for which $A \subseteq F$. Furthermore, it is easy to see that if $A \not\subseteq R(F)$ then this i is unique and does not depend on the choice of F.*

This leads to the following definition.

Definition 2.3.14 (set-valued Ξ function)**.** Let $A \in \mathrm{FIN}(X)$ for each $k > \rho^A$ we define

$$\Xi_A(k) = \begin{cases} -1 & \text{if } A \subseteq R(F) \\ i & \text{if } A \not\subseteq R(F) \text{ and } A \subseteq F_i \end{cases}$$

Here, $F \in \mathcal{F}_k$ is such that $A \subseteq F$.

Remark 2.3.15. It is not hard to check that actually $\Xi_A(k) = \Xi_{\max(A)}(k)$ for each $k > \rho^A$.

The following lemma is easy to prove and it is left to the reader as it is just a variation of Lemma 2.3.9.

Lemma 2.3.16. *For any distinct $A, B \in FIN(X)$ and $k \geqslant \rho^{A \cup B}$ the following happens:*

   (a) *If $0 \leqslant \Xi_A(k) \neq \Xi_B(k) \geqslant 0$ then $k = \rho^{A \cup B}$.*

   (b) *If $k > \rho^{A \cup B}$ then $\Xi_A(k) = \Xi_B(k)$ or $\min(\Xi_A(k), \Xi_B(k)) = -1$.*

2.4 Capturing Axioms

Suppose that A and B are two different elements of $\mathrm{FIN}(X)$ and k is a natural number bigger (or equal) than the diameter of both A and B. Then $(A)_k$ and $(B)_k$ are closed sets by the part (4) of Proposition 2.1.7. By Lemma 2.1.18 and Theorem 2.1.26 we know that these two sets have the same size if and only if they are ρ-isomorphic. However, if this situation occurs, nothing assures that if we take $h : (A)_k \longrightarrow (B)_k$ to be the increasing bijection then $h[A] = B$. In particular, if $k \geqslant \rho^{A \cup B}$, then either $|(A)_k| \neq |(B)_k|$ (so these closures are not even ρ-isomorphic) or $(A)_k = (B)_k$[6]. Note that in the latter case, the increasing bijection h between $(A)_k$ and $(B)_k$ is the identity function which means that $h[A] \neq B$. In virtue of this observation, the biggest k for which we can expect $(A)_k$ and $(B)_k$ to satisfy this stronger isomorphism condition is $\rho^{A \cup B} - 1$.

Definition 2.4.1. Let $A, B \in FIN(X)$ be such that $\rho^A, \rho^B < \rho^{A \cup B}$. We say that A and B are *strongly ρ-isomorphic* if for $l = \rho^{A \cup B}$ it happens that:

   ▪ $|(A)_{l-1}| = |(B)_{l-1}|$. That is, $(A)_{l-1}$ is ρ-isomorphic to $(B)_{l-1}$,

   ▪ $h[A] = B$ where h is the increasing bijection from $(A)_{l-1}$ to $(B)_{l-1}$.

The proof of the following lemma is easy.

Lemma 2.4.2. *If A and B are strongly ρ-isomorphic, then $\rho^A = \rho^B$ and $(A)_k$ is isomorphic to $(B)_k$ for each $k < \rho^{A \cup B}$.*

The following proposition gives us a better picture of how two strongly ρ-isomorphic sets look inside the closure of their union.

Proposition 2.4.3. *Let $A, B \in FIN(X)$ be such that $\max(\rho^A, \rho^B) < \rho^{A \cup B} = l$. Then A and B are strongly ρ-isomorphic if and only if for any $F \in \mathcal{F}_l$ with $A \cup B \subseteq F$ the following happens:*

   ▪ $\Xi_A(l), \Xi_B(l) \neq -1$.

   ▪ *$h[A] = B$ where h is the increasing bijection from $F_{\Xi_A(l)}$ to $F_{\Xi_A(l)}$.*

---

[6]Since $k \geqslant \rho^{A \cup B} \geqslant \max(\rho^A, \rho^B)$, then $(A)_k = (\max(A))_k$ and $(B)_k = (\max(B))_k$.

Proof. *Proof of* $\Leftarrow$. Let $F \in \mathcal{F}_l$ be such that $A \cup B \subseteq F$. Note that $(A)_{l-1} \sqsubseteq F_{\Xi_A(l)}$ and $(B)_{l-1} \sqsubseteq F_{\Xi_B(l)}$. By this and since $|(A)_{l-1}| = |(B)_{l-1}|$ we have that $h[(A)_{l-1}] = (B)_{l-1}$ where $h : F_{\Xi_A(l)} \longrightarrow F_{\Xi_B(l)}$ is the increasing bijection. Hence, the restriction of $h|_{(A)_{l-1}}$ is the increasing bijection from $(A)_{l-1}$ to $(B)_{l-1}$. In this way we conclude that $h[A] = B$. Now suppose towards a contradiction that $\min(\Xi_A(l), \Xi_B(l)) = -1$. Without loss of generality we can assume that $\Xi_A(l) = -1$. That is, $A \subseteq R(F)$. Since $h|_{R(F)}$ is the identity function this means that $A = B$. This is a contradiction to the fact that $\max(\rho^A, \rho^B) < \rho^{A \cup B}$. We conclude that $\min(\Xi_A(l), \Xi_B(l)) \geqslant 0$ so we are done. $\qquad\square$

Proof of $\Rightarrow$. For this, let $F \in \mathcal{F}_l$ be such that $A \cup B \subseteq F$ and $h' : F_{\Xi_A(l)} \longrightarrow F_{\Xi_B(l)}$ be the increasing bijection. Since $h'[A] = B$ it is easy to see that $h'[(A)_{l-1}] = (B)_{l-1}$. In this way, $|(A)_{l-1}| = |(B)_{l-1}|$. To finish just note that $h = h'|_{(A)_{l-1}}$ is the increasing bijection to $(B)_{l-1}$ so $h[A] = B$. $\qquad\square$

$\blacksquare$

Remark 2.4.4. The two conditions imposed to F in the previous proposition are equivalent to the existence of distinct $i, j < n_l$ (namely, $\Xi_A(l)$ and $\Xi_B l$) and a unique $S \subseteq m_{l-1}$ such that $F_i[S] = A$ and $F_j[S] = B$ (such S being $F^{-1}_{\Xi_A(l)}[A]$).

Lemma 2.4.5. *Let* $A, B \in FIN(X)$ *be strongly ρ-isomorphic and let* $l = \rho^{A \cup B}$. *If* $F \in \mathcal{F}_l$ *is such that* $A \cup B \subseteq F$, *then* $A \cap B = A \cap R(F)$. *In particular,* $A \cap B \sqsubseteq A$ *and* $\Xi_{A \cap B}(l) = -1$.

Proof. Let $h : F_{\Xi_A(l)} \longrightarrow F_{\Xi_B(l)}$ be the increasing bijection. The inclusion from left to right is clear since $A \cap B = A \cap (A \cap B) \subseteq A \cap (F_{\Xi_A(l)} \cap F_{\Xi_B(l)}) = A \cap R(F)$. To prove the other one just note that $h|_{R(F)}$ is the identity function. Hence, if $\alpha \in A \cap R(F)$ then $\alpha = h(\alpha) \in B$ due to the second point of Proposition 2.4.3. $\qquad\blacksquare$

As a corollary we have:

Corollary 2.4.6. *Let* $A, B \in FIN(X)$ *be strongly ρ-isomorphic and let* $l = \rho^{A \cup B}$. *If* $F \in \mathcal{F}_l$ *is such that* $A \cup B \subseteq F$ *and* $A \cap B = \emptyset$ *then* $A \subseteq F_{\Xi_A(l)} \backslash R(F)$ *and* $B \subseteq F_{\Xi_A(l)} \backslash R(F)$. *In particular,* $\Xi_\alpha(l) = \Xi_A(l)$ *and* $\Xi_\beta(l) = \Xi_B(l)$ *for any* $\alpha \in A$ *and* $\beta \in B$.

Remark 2.4.7. In virtue of Proposition 2.4.3, if $A = \{\alpha\}$ and $B = \{\beta\}$ for α and β distinct ordinals, then A and B are strongly ρ-isomorphic if and only if $\Delta(\alpha, \beta) = \rho(\alpha, \beta)$.

Definition 2.4.8 (Captured families)**.** Let $\mathcal{D}$ be a finite subset of $FIN(X)$ and $l \in \omega$. We say that that $\mathcal{D}$ is *captured* at level l if $2 \leqslant |\mathcal{D}| \leqslant n_l$ and:

(1) $l = \rho^{\cup \mathcal{D}} > \rho^D$ for each $D \in \mathcal{D}$,

(2) $\{\Xi_D(l) : D \in \mathcal{D}\} = \{0, \ldots, |\mathcal{D}| - 1\} = |\mathcal{D}|.$[7]

---

[7]by Lemma 2.3.16, this condition implies that $\rho^{E \cup D} = l$ for any distinct $D, E \in \mathcal{D}$.

(3) For any distinct $D, E \in \mathcal{D}$, D is strongly ρ-isomorphic to E.

Equivalently, for any $F \in \mathcal{F}_l$ with $\bigcup \mathcal{D} \subseteq F$ there is $S \subseteq m_{l-1}$ such that

$$\mathcal{D} = \{F_i[S] \ : \ i < |\mathcal{D}|\}.$$

Additionally, if $|\mathcal{D}| = n_l$ we say that $\mathcal{D}$ is *fully captured* at level l. Whenever $D \in FIN(X)$, we say that D is captured (resp. fully captured) in case $\{\{\alpha\} \ : \ \alpha \in D\}$ is captured (resp. fully captured). Finally, if $\mathcal{D}$ is captured at level l and we write $\mathcal{D}$ as a list of elements, say $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$, we always assume that $\Xi_{D_i}(l) = i$ for each $i < n$.
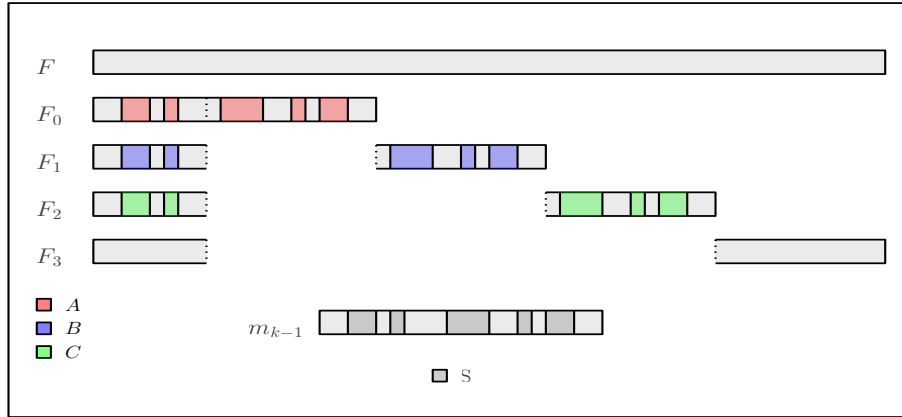
In here, the set $\mathcal{D} = \{A, B, C\}$ is being captured as testified by F and S.
Note that $\mathcal{D}$ forms a Δ-system whose root is contained in $R(F)$.

We will frequently use the following lemma without any explicit mention to it. Its proof is left to the reader.

Lemma 2.4.9. *Let $\mathcal{D}$ be a finite subset of $FIN(X)$ which is captured at some level $l \in \omega$. If n is the common cardinality of the members of $\mathcal{D}$ and S is a non-empty finite subset of n, then*

$$\{D[S] \ : \ D \in \mathcal{D}\}$$

is also captured at level l as long as this set has at least two elements.

The following proposition is a direct consequence of the Remark 2.4.7.

Proposition 2.4.10. *Let $l \in \omega$ and $D \in FIN(X)$ be such that $|D| \geqslant 2$. Then D is captured at level l if and only if:*

(1) *For any distinct $\alpha, \beta \in D$, $\Delta(\alpha, \beta) = l = \rho(\alpha, \beta)$,*

(2) *For each $i < |D|$, $\Xi_{D(i)}(l) = i$. In other words, $\{\Xi_\alpha(l) \ : \ \alpha \in D\} = |D|$.*

Lemma 2.4.11. *Let $\mathcal{D}$ be a finite subset of $FIN(X)$ and $l \in \omega$. For any $A \in \mathcal{D}$, let $\alpha_A = \max(A)$. Suppose that there are $j \in \omega$, $a < m_j$ and $C \subseteq a + 1$ so that the following conditions hold for any $A \in \mathcal{D}$:*

(a) $\rho^A \leqslant j$,

(b) $\|\alpha_A\|_j = a$,

(c) $(\alpha_A)_j[C] = A$.

If $D = \{\alpha_A : A \in \mathcal{D}\}$ is captured at some level l, then $\mathcal{D}$ is also captured at level l.

Proof. First observe that that $l > j$. In order to prove that $\mathcal{D}$ is captured at level l, we will prove that the three points in Definition 2.4.8 are satisfied.

<u>Claim 1</u>: $l = \rho^{\cup\mathcal{D}}$ and $l > \rho^A$ for each $A \in \mathcal{D}$.

Proof of claim. We know that $l > \rho^A$ for each $A \in \mathcal{D}$ due to the condition (a) written above. $l \leqslant \rho^{\cup\mathcal{D}}$ because if $A, B \in \mathcal{D}$ are distinct then $\alpha_A, \alpha_B \in \bigcup\mathcal{D}$ and $\rho(\alpha_A, \alpha_B) = l$. Lastly, if $F \in \mathcal{F}_l$ is such that $D \subseteq F$ then $A \subseteq (\alpha_A)_j \subseteq (\alpha_A)_l \subseteq F$ for any $A \in \mathcal{D}$. In this way, $\bigcup\mathcal{D} \subseteq F$. Thus, $\rho^{\mathcal{D}} \leqslant l$. $\square$

<u>Claim 2</u>: $\{\Xi_A(l) : A \in \mathcal{D}\} = n_l$.

Proof of claim. As $l > \rho^A$ for each $A \in \mathcal{D}$, then $\Xi_A(l) = \Xi_{\alpha_A}(l)$ for any $A \in \mathcal{D}$ due to the Remark 2.3.15. The claim follows from this fact. $\square$

<u>Claim 3</u>: For any distinct $A, B \in \mathcal{D}$, A is strongly ρ-isomorphic to B.

Proof of claim. Let $h : (A)_{l-1} \implies (B)_{l-1}$ be the increasing bijection. As D is fully captured, we know that $\{\alpha_A\}$ is strongly isomorphic to $\{\alpha_B\}$. In this way, $h(\alpha_A) = \alpha_B$. Since $k \leqslant l - 1$, we conclude that $h[(\alpha_A)_k] = (\alpha_B)_k$. Thus,

$$h[A] = h[\,(\alpha_A)_k[C]\,] = (\alpha_B)_k[C] = B.$$

This finishes the proof. $\square$

$\blacksquare$

Definition 2.4.12 (*n*-capturing schemes)**.** Let $\mathcal{P}$ be a partition of ω and $n \in \omega$. We say that $\mathcal{F}$ is *n-$\mathcal{P}$-capturing* if for each uncountable $\mathcal{S} \subseteq \mathrm{FIN}(X)$ and $P \in \mathcal{P}$ there are infinitely many $l \in P$ with some $\mathcal{D} \in [\mathcal{S}]^n$ which is captured at level l. Whenever $\mathcal{P} = \{\omega\}$, we simply say that $\mathcal{F}$ is *n-capturing*.

Definition 2.4.13 (capturing schemes)**.** Let $\mathcal{P}$ be a partition of ω. We say that $\mathcal{F}$ is *$\mathcal{P}$-capturing* (resp. *capturing*) if $\mathcal{F}$ is *n-$\mathcal{P}$*-capturing (resp. *n-capturing*) for each $n \in \omega$.

Definition 2.4.14 (fully capturing schemes)**.** Let $\mathcal{P}$ be a partition of ω. We say that $\mathcal{F}$ is *$\mathcal{P}$-fully capturing* if for each uncountable $\mathcal{S} \subseteq \mathrm{FIN}(X)$ and $P \in \mathcal{P}$ there are infinitely many $l \in P$ with some $\mathcal{D} \in \mathrm{FIN}(\mathcal{S})$ which is fully captured at level l. Whenever $\mathcal{P} = \{\omega\}$, we simply say that $\mathcal{F}$ is *fully capturing*.

The following Lemma was first proved in [112] (Lemma 7.1) and it presents useful equivalences of the previous definitions.

Lemma 2.4.15. *Let $\mathcal{F}$ be a construction scheme and $\mathcal{P}$ be a partition of ω compatible with τ. Then:*

- *For each $n \in \omega$, $\mathcal{F}$ is n-$\mathcal{P}$-capturing if and only if for each $S \in [X]^{\omega_1}$ and $P \in \mathcal{P}$ there is $D \in [S]^n$ which is captured at some level $l \in P$.*

- *$\mathcal{F}$ is $\mathcal{P}$-fully capturing if and only if for each $S \in [X]^{\omega_1}$ and $P \in \mathcal{P}$ there is $D \in FIN(S)$ which is fully captured at some level $l \in P$.*

Proof. We will only prove the second point as the first one is proved in a similar way. In order to prove the nontrivial direction of this statement, let $\mathcal{S}$ be an uncountable subset of $FIN(X)$ and fix $k \in \omega$. For any $A \in S$, let $\alpha_A = \max(A)$. Due to the pigeonhole principle, we can find an uncountable $\mathcal{S}' \subseteq S$, $k < j \in \omega$, $a < m_j$ and $C \subseteq a$ so that the following conditions hold for any $A \in \mathcal{S}'$:

(a) $\rho^A \leqslant j$.

(b) $\|\alpha_A\|_j = a$,

(c) $(\alpha_A)_j[C] = A$.

By the point (b), it follows that $\rho(\alpha_A, \alpha_B) > j$ for any two distinct $A, B \in \mathcal{S}'$. According to our assumptions, there is $\mathcal{D} \in FIN(\mathcal{S})$ for which $D = \{\alpha_A : A \in \mathcal{D}\}$ is fully captured at some level l. Then $\mathcal{D}$ is captured in level l by virtue of the Lemma 2.4.11. Thus, the proof is over.

∎

We are now ready to state the Capturing axioms. The axiom $FCA(part)$ was introduced in [112]. The other axioms were later studied in [47], [58], [68] and [69].

Fully Capturing Axiom [FCA]: There is a fully capturing construction scheme over ω_1 of every possible good type.

Fully Capturing Axiom with Partitions [FCA(part)]: There is a $\mathcal{P}$-fully capturing construction scheme over ω_1 for every good type τ and each partition $\mathcal{P}$ compatible with τ.

n-Capturing Axiom [CA$_n$]: For any $n' \leqslant n$, there is an n'-capturing construction scheme over ω_1 of every possible good type satisfying that $n' \leqslant n_k$ for each $k \in \omega \backslash 1$.

n-Capturing Axiom with Partitions [CA$_n$(part)]: For any $n' \leqslant n$, there is a $\mathcal{P}$-n'-capturing construction scheme over ω_1 for every good type τ satisfying that $n' \leqslant n_k$ for each $k \in \omega \backslash 1$ and each partition $\mathcal{P}$ compatible with τ.

Capturing Axiom [CA]: CA_n holds for each $n \in \omega$ and there is a capturing construction scheme over ω_1 for every good type satisfying that the sequence $\langle n_{k+1} \rangle_{k \in \omega}$ is non-decreasing and unbounded.

Capturing Axiom with partitions [CA(part)]: $CA_n(part)$ holds for each $n \in \omega$ and there is a $\mathcal{P}$-capturing construction scheme over ω_1 for every good type τ satisfying that the sequence $\langle n_{k+1} \rangle_{k \in \omega}$ is non-decreasing and unbounded and each partition $\mathcal{P}$-compatible with τ.

Note that the following relations hold between the previously defined capturing axioms where the arrows denote the implication relation.

$$CA_2(part) \Longleftarrow CA_3(part) \Longleftarrow \ldots \Longleftarrow CA(part) \Longleftarrow FCA(part)$$
$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$
$$CA_2 \Longleftarrow CA_3 \Longleftarrow \ldots \Longleftarrow CA \Longleftarrow FCA$$

In [58], Damian Kalajdzievski and Fulgencio Lopez proved that none of the arrows going from CA_{n+1} to CA_n (resp. from $CA_{n+1}(part)$ to $CA_n(part)$) can be reversed. They also proved the following theorem.

Theorem 2.4.16. *Let κ be an uncountable cardinal. Then $\mathbb{1}_{\mathbb{C}_\kappa} \vDash$ " $FCA(part)$ ".*

For the sake of completeness, we will prove such results later on. Furthermore, we will show that it is consistent that there is an n-capturing construction scheme which is not $\mathcal{P}$-n-capturing for any partition of ω in at least two infinite pieces. For this, we need to introduce a new cardinal invariant.

Definition 2.4.17 (Parametrized Martin's numbers). Let $\mathcal{F}$ be a construction scheme and $2 \leqslant n \in \omega$. We define $\mathfrak{m}_{\mathcal{F}}^n$ as follows:

$$\mathfrak{m}_{\mathcal{F}}^n = \begin{cases} \omega & \text{if } \mathcal{F} \text{ is not } n\text{-capturing} \\ \min(\mathfrak{m}(\mathbb{P}) \ : \ \mathbb{P} \text{ is ccc and } \mathbb{P} \Vdash \text{``} \mathcal{F} \text{ is } n - capturing\text{''}) & \text{if } \mathcal{F} \text{ is } n\text{-capturing} \end{cases}$$

$\mathfrak{m}_{\mathcal{F}}^2$ is denoted simply as $\mathfrak{m}_{\mathcal{F}}$.

Of course, if $\mathcal{F}$ is n-capturing then $\mathfrak{m}_{\mathcal{F}}^n \geqslant \omega_1$. The following theorem will be proved in Chapter 5.

Theorem 2.4.18. *For any n-capturing construction scheme $\mathcal{F}$ there is a ccc-forcing $\mathbb{P}$ for which*
$$\mathbb{P} \Vdash \text{``} \mathfrak{m}_{\mathcal{F}}^n > \omega_1 \text{''}.$$

We end this chapter by announcing the main theorem of the thesis.

Theorem 2.4.19. *The $\Diamond$-principle implies $FCA(part)$.*

Chapter 3

Applications

3.1   Gaps, towers and almost disjoint families

This section is dedicated to the study of gaps and towers. In a general framework, one of the main reasons for studying gaps in Boolean algebras is because of their close relation with the existence of extensions of both homomorphisms and embeddings from one Boolean algebra to another one. More concretely, in [97] Roman Sikorski implicitly gave a criterion for the extension of homomorphisms in Boolean algebras. Such criterion tells us that the objects the we now know as gaps are the only thing which can prevent an homomorphism to be extended. As we will center our attention solely on gaps over $\mathscr{P}(\omega)/\text{FIN}$, all the definitions included here are stated only for such algebra.

The reader interested in knowing more about towers and gaps is referred to [13], [14], [37], [80], [91], [102], [110], [113], [121], [122] and [132].

We start by recalling the main definitions regarding this chapter.

Definition 3.1.1 (Towers)**.** Let X be a countable set and $\mathcal{T} \subseteq [X]^\omega$. We say that $\mathcal{T}$ is a *tower*[1] if it is well-ordered with respect to $\subseteq^*$. Furthermore, for an ordinal κ, we say that $\mathcal{T}$ is a κ-tower whenever it is a tower and $ot(\mathcal{T}) = \kappa$.

Definition 3.1.2 (Gaps and Pregaps)**.** Let X be a countable set and $\mathcal{L}, \mathcal{R} \subseteq [X]^\omega$. We say that $(\mathcal{L}, \mathcal{R})$ is *pregap*, and write it as $\mathcal{L} \perp \mathcal{R}$, if $L \cap R =^* \emptyset$ for all $L \in \mathcal{L}$ and $R \in \mathcal{R}$. An element $C \in [X]^\omega$ is said to *separate* $(\mathcal{L}, \mathcal{R})$ if $L \subseteq^* C$ and $C \cap R =^* \emptyset$ for each $L \in \mathcal{L}$ and $R \in \mathcal{R}$. Finally, we say that $(\mathcal{L}, \mathcal{R})$ is a *gap* if it is a pregap and there is no $C \in [X]^\omega$ separating it.

Remark 3.1.3. It is well known that there are no countable gaps. That is, there is no gap $(\mathcal{L}, \mathcal{R})$ with both $\mathcal{L}$ and $\mathcal{R}$ countable.

Remark 3.1.4. Whenever the domain of a pregap is not specified (the set X of the previous definition) we assume that such domain is ω.

---

[1]It is also common in the literature to call these objects pre-towers, and towers are pre-towers which are maximal with respect to the end-extension. As we won't be dealing with maximal pre-towers, we've opted to simply refer to them as "towers" to simplify the notation.

Definition 3.1.5 (Type of a gap)**.** Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be two partial orders. We say that a pregap $(\mathcal{L}, \mathcal{R})$ is an (X, Y)-*pregap* if $(\mathcal{L}, \subseteq^*)$ and $(\mathcal{R}, \subseteq^*)$ are isomorphic to $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ respectively. Furthermore, if $(\mathcal{L}, \mathcal{R})$ is a gap we will refer to it as an (X, Y)-gap. Whenever both sides of a pregap $(\mathcal{L}, \mathcal{R})$ are index by some set I, say $\mathcal{L} = \langle A_i \rangle_{i \in I}$ and $\mathcal{R} = \langle B_i \rangle_{i \in I}$ we may refer to $(\mathcal{L}, \mathcal{R})$ as $(A_i, B_i)_{i \in I}$. Of course, if I is a partial order and we specify that the pregap is an (I, I)-gap it is understood such an indexing is order preserving.

It is easy to construct an (ω_1, ω_1)-gap using CH. However, in 1909, Felix Hausdorff gave a clever recursive construction of an (ω_1, ω_1)-gap without appealing to any extra axioms. This is quite surprising, as it is required to overcome $\mathfrak{c}$ obstacles in only ω_1-many steps. In Theorem 3.1.21, we will prove an analogous result but for large collection of partial orders of size ω_1.

The gap that Hausdorff constructed satisfied the following property.

Definition 3.1.6 (Hausdorff condition)**.** Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap on ω. We say that $(\mathcal{L}, \mathcal{R})$ is Hausdorff if $\{R \in \mathcal{R} : rank(R) < rank(L) \ and \ L \cap R \subseteq k\}$ is finite for each $L \in \mathcal{L}$ and $k \in \omega$.

Definition 3.1.7 (Luzin condition)**.** Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap on an infinite set X. We say that $(\mathcal{L}, \mathcal{R})$ is Luzin if $\{R \in \mathcal{R} : rank(R) < rank(L) \ and \ |L \cap R| \leqslant k\}$ is finite for each $L \in \mathcal{L}$ and $k \in \omega$.

Remark 3.1.8. Note that the Luzin condition is stronger than the Hausdroff condition. It is a standard excercise to prove that any (ω_1, ω_1)-pregap satisfying the Hausdorff condition is in fact a gap. Under the P-ideal Dichotomy (PID), if $(\mathcal{L}, \mathcal{R})$ is an (ω_1, ω_1)-gap then there are cofinal $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{R}' \subseteq \mathcal{R}$ so that $(\mathcal{L}', \mathcal{R}')$ is a Hausdorff gap. This result was proved by Uri Abraham and Stevo Todorčević in [4]. The same result follows from $\mathfrak{m} > \omega_1$.

In the following theorem we construct a Hausdorff gap using a 2-construction scheme (that is, a construction scheme of type $\langle m_k, 2, r_{k+1} \rangle_{k \in \omega}$). The first construction of a Hausdorff gap using morasses was performed by Daniel Velleman in [127]. In [69] and [110] the reader may find a construction of such a gap using construction schemes and ordinal metrics respectively. First, a quick remark.

Remark 3.1.9. If $\langle L_\alpha \rangle_{\alpha \in \omega_1}$ and $\langle R_\alpha \rangle_{\alpha \in \omega_1}$ are ω_1-towers and $L_\alpha \cap R_\alpha =^* \emptyset$ for each $\alpha \in \omega_1$ then $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is a pregap. This is because for each $\alpha < \beta \in \omega_1$, $L_\alpha \cap R_\beta \subseteq^* L_\beta \cap R_\beta$.

Theorem 3.1.10. *Let $\mathcal{F}$ be a 2-construction scheme. For each $\alpha \in \omega_1$, define*

$$L_\alpha = \{2k + \Xi_\alpha(k) : k \in \omega \backslash 1, \ and \ \Xi_\alpha(k) \geqslant 0\},$$

$$R_\alpha = \{2k + (1 - \Xi_\alpha(k)) : k \in \omega \backslash 1, \ and \ \Xi_\alpha(k) \geqslant 0\}.$$

Then, $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is a Hausdorff gap.

Proof. Since $r_k = 0$ for infinitely many $k's$, it should be clear that each L_α and R_α are infinite. By virtue of the part (c) in Lemma 2.3.9, we have that if $\alpha < \beta$ then

$$L_\alpha \backslash L_\beta \subseteq \{2k + \Xi_\alpha(k) : k \leqslant \rho(\alpha, \beta) \ and \ \Xi_\alpha(k) \neq \Xi_\beta(k)\},$$

$$R_\alpha \backslash R_\beta \subseteq \{2k + (1 - \Xi_\alpha(k)) \, : \, k \leqslant \rho(\alpha, \beta) \; and \; \Xi_\alpha(k) \neq \Xi_\beta(k)\}.$$

As the sets on the right are finite, this means $\langle L_\alpha \rangle_{\alpha \in \omega_1}$ and $\langle R_\alpha \rangle_{\alpha \in \omega_1}$ are both ω_1-towers. Furthermore by definition we have that $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha \in \omega_1$. This implies that $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is a pregap due to the Remark 3.1.9.

The only thing left to prove is that the Hausdorff condition is satisfied. For this purpose take $\beta \in \omega_1$ and $k \in \omega$. We claim $\{\alpha < \beta \, : \, L_\beta \cap R_\alpha \subseteq k\} \subseteq (\beta)_k$. For this, take an arbitrary $\alpha < \beta$ satisfying $\rho(\alpha, \beta) \geqslant k$. By the part (b) of Lemma 2.3.9, $\Xi_\alpha(\rho(\alpha, \beta)) = 0$ and $\Xi_\beta(\rho(\alpha, \beta)) = 1$. This means $2\rho(\alpha, \beta) + 1 \in L_\beta \cap R_\alpha$, so we are done. ∎

An interesting feature of the proof given above is that not only we prove that there is a gap using a 2-construction scheme, but such gap can be explicitly defined from it.

For a countably infinite set X, we say that a family $\mathcal{A} \subseteq [X]^\omega$ is an almost disjoint family if $A \cap B =^* \emptyset$ whenever $A, B \in \mathcal{A}$ are different. Almost disjoint families are one of the central objects of study in modern combinatorial set theory. Constructing almost disjoint families with special properties is usually difficult and in most cases had lead to the development of powerful tools (see [6], [15], [21], [22], [31], [38],[84] and [96]). Almost disjoint families have also played a central roll in the solution of many problems of Topology and Analysis. An example of this is the solution of the selection problem posed by Jan van Mill and Evert Wattel in [126] and solved by Michael Hrušák and Iván Martínez-Ruíz in [50]. The reader interested in learning more about almost disjoint families is refered to [46], [48] and [49].

Our next goal is to use almost disjoint families with the aim of proving that, basically, there are gaps of any possible type for which the cofinality of the two partial ordered sets involved is ω_1.

Definition 3.1.11. Let $\mathcal{A}$ be an almost disjoint family of size ω_1 over a set X. We say that:

- $\mathcal{A}$ is *inseparable* if for any two disjoint $\mathcal{L}, \mathcal{R} \in [\mathcal{A}]^{\omega_1}$, the pair $(\mathcal{L}, \mathcal{R})$ forms a gap.

- $\mathcal{A}$ is *Luzin* if we can enumerate it as $\langle A_\alpha \rangle_{\alpha \in \omega_1}$ in such way that $\{\alpha \in \beta \, : \, |A_\alpha \cap A_\beta| < n\}$ is finite for each $\beta \in \omega_1$ and $n \in \omega$.

- $\mathcal{A}$ is *Jones* if for any two disjoint $\mathcal{L} \in [\mathcal{A}]^\omega$ and $\mathcal{R} \in [\mathcal{A}]^{\omega_1}$, the pregap $(\mathcal{L}, \mathcal{R})$ can be separated. If $\mathcal{A}$ is indexed as $\langle A_\alpha \rangle_{\alpha \in \omega_1}$ then $\mathcal{A}$ is Jones if and only if for any $\beta \in \omega_1$ the pregap $(\langle A_\alpha \rangle_{\alpha \leqslant \beta}, \langle A_\delta \rangle_{\delta > \beta})$ can be separated.

- $\mathcal{A}$ is *Luzin-Jones* if it is both Luzin and Jones.

Remark 3.1.12. It is easy to check that any Luzin family is in fact inseparable. In [71], Judith Roitman and Lajos Soukup showed that under $\mathfrak{m} > \omega_1$ any uncountable AD family either contains a Luzin family or contains no inseparable family.

The first construction of a Luzin family was done in [71] by Nikolái Nikoláyevich Luzin. A Jones family was implicitly constructed by F. B. Jones in [57]. Although, at first glance, Luzin and Jones properties seem to be incompatible, a construction of a Luzin-Jones family was obtained in [42] by Osvaldo Guzmán , Michael Hrušák and

Piotr Koszmider (building from work by Koszmider in [63]). We would like to point out that the highly complex construction of the mentioned family is carried out through the use of simplified morasses. Here we give an elementary construction of such object.

Theorem 3.1.13. *There is a Luzin-Jones family.*

Proof. Let $\mathcal{F}$ be a 2-construction scheme. For every $k \in \omega \backslash 1$, let

$$N_k = \{k\} \times (m_{k-1} \backslash r_k) \times [k \cdot m_{k-1},\, k \cdot m_k).$$

We aim to construct a Luzin-Jones family $\langle A_\alpha \rangle_{\alpha \in \omega_1}$ over the union of the N_k's, namely N. For this, take an arbitrary $\alpha \in \omega_1$. Given $k \in \omega \backslash 1$, we define $A_\alpha^k \subseteq N_k$ as follows:

(a) If $\Xi_\alpha(k) = -1$, let $A_\alpha^k = \emptyset$.

(b) If $\Xi_\alpha(k) = 0$, let $A_\alpha^k = \{k\} \times \{\|\alpha\|_k\} \times [k \cdot m_{k-1},\, k \cdot m_k)$.

(c) If $\Xi_\alpha(k) = 1$, let $A_\alpha^k = \{k\} \times [r_k, m_{k-1}) \times \left[k \cdot \|\alpha\|_k,\, k(\|\alpha\|_k + 1)\right)$.

Now, we define A_α as $\bigcup_{k \in \omega \backslash 1} A_\alpha^k$ and $\mathcal{A}$ as $\langle A_\alpha \rangle_{\alpha \in \omega_1}$.

Our first task is to show that $\mathcal{A}$ is an almost disjoint family. First observe that since $r_k = 0$ infinitely often it follows that each A_α is in fact infinite. Now let $\alpha < \beta \in \omega_1$ and take an arbitrary $k > \rho(\alpha, \beta)$. In virtue of the part (c) by Lemma 2.3.9 we have that $\Xi_\alpha(k) = -1$ or $\Xi_\alpha(k) = \Xi_\beta(k)$. In either case it follows that $A_\alpha^k \cap A_\beta^k = \emptyset$. Therefore we conclude that

$$A_\alpha \cap A_\beta = \bigcup_{k \in \omega \backslash 1} \left(A_\alpha^k \cap A_\beta^k \right) \subseteq \bigcup_{k \leqslant \rho(\alpha, \beta)} N_k.$$

Now we prove that $\mathcal{A}$ is Luzin. We claim that whenever $\alpha < \beta$ and $k = \rho(\alpha, \beta)$ then $|A_\alpha \cap A_\beta| \geqslant k$. If this happens then $\{\alpha < \beta : |A_\alpha \cap A_\beta| < k\} \subseteq (\beta)_k$ for any $\beta \in \omega_1$. Let α, β and k be as previously stated. The part (b) of Lemma 2.3.9 assures $\Xi_\alpha(k) = 0 < \Xi_\beta(k) = 1$. In this way

$$A_\alpha^k \cap A_\beta^k = \{k\} \times \{|(\alpha)_k^-|\} \times \left[k \cdot \|\alpha\|_k,\, k(\|\alpha\|_k + 1)\right).$$

The cardinality of this set is k and it is contained in $A_\alpha \cap A_\beta$. Thus, we are done.
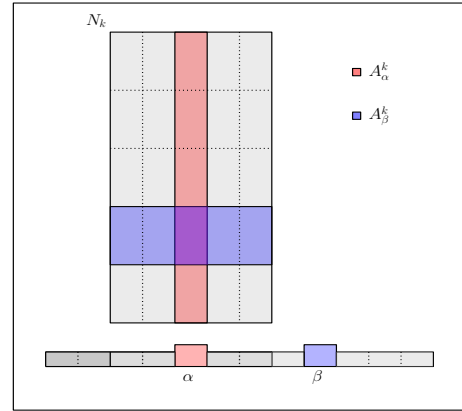
In here, $\Xi_\alpha(k) = 0$ and $\Xi_\beta(k) = 1$.

We will end the proof by showing that $\mathcal{A}$ is a Jones family. For this purpose define C_β as

$$\bigcup_{k \in \omega} \left(\bigcup_{\alpha \in (\beta)_k} A_\alpha^{k+1} \right)$$

for each $\beta \in \omega_1$. We will show that C_β separates $(\langle A_\alpha \rangle_{\alpha \leqslant \beta}, \langle A_\alpha \rangle_{\delta > \beta})$. Indeed, if $\alpha \leqslant \beta$ then $A_\alpha^{k+1} \subseteq C_\beta$ for each $k \geqslant \rho(\alpha, \beta)$, so in particular $A_\alpha \subseteq^* C_\beta$. On the other hand,

if $\delta > \beta$ then $A_\delta^{k+1} \cap A_\alpha^{k+1} = \emptyset$ for any $k \geqslant \rho(\delta, \beta)$ and each $\alpha \in (\beta)_k$. This is because for any such α we have that $\rho(\alpha, \delta) \leqslant k$. Consequently $A_\delta \cap C_\beta =^* \emptyset$. This finishes the proof. ∎

The previous construction yields an almost disjoint family which, in principle, satisfies a lot more properties than just being Luzin-Jones. In the rest this section, we will analyse those properties. The following definitions are essential for that task.

Definition 3.1.14 (Succesor-like elements)**.** Let $(X, \leqslant)$ be a partial order. We say that $x \in X$ is *succesor-like* if:

- $pred(x)$ is finite,

- for any $y < x$ there is $z \in pred(x)$ for which $y \leqslant z$.

Definition 3.1.15 (Luzin representation)**.** Let $(X, \leqslant)$ be a partial order of size ω_1. A *Luzin representation of X* is an ordered pair $(\mathcal{T}, \mathcal{A})$ consisting of two families of infinite subsets of ω indexed as $\langle T_x \rangle_{x \in X}$ and $\langle A_x \rangle_{x \in X}$ respectively. Furthermore, $\mathcal{A}$ is a Luzin family and for all $x, y \in X$ the following properties hold:

(a) $A_x \subseteq T_x$.

(b) If $y \nleqslant x$, then $A_y \subseteq^* T_y \backslash T_x$.

(c) If $\inf(x, y)$ exists, then $T_x \cap T_y =^* T_{\inf(x,y)}$.

(d) If x is succesor-like, then

$$T_x \backslash \left(\bigcup_{z \in pred(x)} T_z \right) =^* A_x.$$

(e) If x and y are incompatible, then $T_x \cap T_y =^* \emptyset$.

In particular $(\mathcal{T}, \subseteq^*)$ is order isomorphic to X by virtue of the points (b) and (c). In the case there is $(\mathcal{T}, \mathcal{A})$ a Luzin representation of X we say that $\mathcal{A}$ *codes X*.

Definition 3.1.16 (ω_1-like order)**.** We say that a partial order $(X, \leqslant)$ is ω_1-like if it is well-founded, $|X| = \omega_1$ and $|(-\infty, x)| \leqslant \omega$ for all $x \in X$.

Proposition 3.1.17. *Let* $(Y, \leqslant)$ *be a partial order of cardinality* ω_1. *There is a well-founded cofinal* $X \subseteq Y$ *with* $|(-\infty, x)_X| \leqslant \omega$ *for each* $x \in X$.

Proof. Enumerate Y as $\langle y_\alpha \rangle_{\alpha \in \omega_1}$. Define X as the set of all y_β's such that $y_\beta \nleqslant y_\alpha$ for each $\alpha < \beta$. Of course X is well-founded as for each $x, z \in X$ with $x = y_\alpha$ and $z = y_\beta$ it happens that if $x < z$ then $\alpha < \beta$. Because of this we also have that $(-\infty, y_\beta)_X \subseteq \{y_\alpha : \alpha < \beta\}$ for each $y_\beta \in X$. Lastly X is cofinal in Y because for each $\beta \in \omega_1$ the element $y_{\xi_\beta} \in X$ where $\xi_\beta = \min(\alpha \leqslant \beta : y_\alpha \geqslant y_\beta)$. ∎

The following lemma is easy.

Lemma 3.1.18. *Let $(X, \leqslant)$ be ω_1-like. Then there is a bijection $\phi : X \longrightarrow \omega_1$ with $\phi(x) < \phi(y)$ for all $x < y \in X$.*

Let $(X, \leqslant)$ ω_1-like and $\phi : X \longrightarrow \omega_1$ as above. In the next lemma, in some sense, we will pull back a construction scheme from ω_1 to some sort of "construction scheme" on X. This suggests that the theory of construction schemes could be generalized to other partial orders of size ω_1. The extend and utility of this idea remains mainly unexplored.

Lemma 3.1.19. *Let $\mathcal{F}$ be a 2-construction scheme and let X and ϕ be as in Lemma 3.1.18. For each $x \in X$ and $k \in \omega$ define $M_x^k = \{\, z \leqslant x \, : \, \phi(z) \in (\phi(x))_k \,\}$. The following properties hold for each $x, y \in X$ and $k \in \omega$:*

(1) *If $\inf(x,y)$ exists and $k > \rho^{\phi[\{x,y,\inf(x,y)\}]}$, then $M_x^k \cap M_y^k = M_{\inf(x,y)}^k$.*

$(\frac{3}{2})$ *If $y \leqslant x$ and $k > \rho^{\phi[\{x,y\}]}$, then $M_y^k \subseteq M_x^k$.*

(2) *If $y \nleqslant x$, then $y \in M_y^k \backslash M_x^k$.*

(3) *If x is succesor-like and $k > \rho^{\phi[pred(x) \cup \{x\}]}$, then $M_x^k \backslash \Big(\bigcup_{z \in pred(x)} M_z^k \Big) = \{x\}$.*

(4) *If x and y are incompatible, then $M_x^k \cap M_y^k = \emptyset$.*

Proof. The points (2) and (4) are trivial and the point $(\frac{3}{2})$ follows directly from (1). Therefore, we will only prove (1) and (3).

Proof of (1). Since ϕ preserves the order then $\phi(\inf(x,y)) \leqslant \min(\phi(x), \phi(y))$ which implies that $(\phi(\inf(x,y))_k \sqsubseteq (\phi(x))_k \cap (\phi(y))_k$. In this way, if $z \in M_{\inf(x,y)}^k$ then $z \leqslant x, y$ by definition of the infimum, and z belongs to both $(\phi(x))_k$ and $(\phi(y))_k$ because $z \in (\phi(\inf(x,y)))_k$. Therefore $z \in M_x^k \cap M_y^k$. On the other hand, if $z \in M_x^k \cap M_y^k$ then $z \leqslant x$ and $z \leqslant y$, so $z \leqslant \inf(x,y)$. Furthermore, $\phi(z) \in (\phi(x))_k \cap (\phi(y))_k$ and $\phi(z) \leqslant \phi(\inf(x,y))$. As $(\phi(\inf(x,y)))_k$ is an initial segment of such intersection then $\phi(z)$ belongs to it. Consequently $z \in M_{\inf(x,y)}^k$. $\qquad\square$

Proof of (3). Note that since $x \nleqslant z$ for any $z \in pred(x)$ then $x \in M_x^k \backslash M_z^k$ for any such z. This proves the inclusion from right to left. To show that the one from left to right also holds, let

$$w \in M_x^k \backslash \Big(\bigcup_{z \in pred(x)} M_z^k \Big).$$

Suppose towards a contradiction that $w < x$ and consider $z \in pred(x)$ with $w \leqslant z$. Since $k > \rho(\phi(x), \phi(z))$ we conclude that $(\phi(z))_k \sqsubseteq (\phi(x))_k$. But then $\phi(w) \in (\phi(z))_k$ because $\phi(w) \leqslant \phi(z)$ and $\phi(w) \in (\phi(x))_k$. Consequently $w \in M_z^k$ which is a contradiction. This finishes the proof. $\qquad\square$

∎

In the next theorem we show that not only a great variety of partial orders posses a Luzin representation, but that the particular Luzin family that we constructed in the Theorem 3.1.13 codes all of them. For that reason, it will be convenient to recall some basic facts regarding such family.

We have a countable set N and $\langle N_k \rangle_{k \in \omega}$ a partition of N into finite sets. For each $\alpha \in \omega_1$ we have $A_\alpha^k \subseteq N_k$ and we defined $A_\alpha = \bigcup_{k \in \omega} A_\alpha^{k+1}$ and $\mathcal{A} = \langle A_\alpha \rangle_{\alpha \in \omega_1}$. One key feature of $\mathcal{A}$ is that:

If $\alpha, \beta \in \omega_1$ are distinct and $k > \rho(\alpha, \beta)$ then $A_\alpha^k \cap A_\beta^k = \emptyset$.

Theorem 3.1.20. *Let $\mathcal{A}$ be the Luzin family constructed in Theorem 3.1.13 and let $(X, \leqslant)$ an ω_1-like order. Then there is a family $\mathcal{T} = \langle T_x \rangle_{x \in X} \subseteq [\omega]^\omega$ and a re-indexing of $\mathcal{A}$ as $\langle \hat{A}_x \rangle_{x \in X}$ so that $(\mathcal{T}, \mathcal{A})$ is a Luzin representation of X.*

Proof. Let $\phi : X \longrightarrow \omega_1$ be as in Lemma 3.1.18 and M_x^k be as in Lemma 3.1.19 for any $x \in X$ and $k \in \omega$. Fix $x \in X$ and let $\hat{A}_x = A_{\phi(x)}$. This defines the re-indexing of $\mathcal{A}$. Now, for each $k \in \omega$ and $x \in X$, let

$$T_x^k = \bigcup \{ A_{\phi(z)}^{k+1} : z \in M_x^k \} = \bigcup \{ A_\xi^{k+1} : \xi \in \phi[M_x^k] \}$$

$$T_x = \bigcup_{k \in \omega} T_x^k.$$

We claim that $(\mathcal{T}, \mathcal{A})$ is a Luzin representation of X. In the following paragraphs we will show that the points (a), (b), (c), (d) and (e) of Definition 3.1.15 are satisfied for such pair.

Proof of (a). Given $k > 0$ we have that $A_{\phi(x)}^k \subseteq T_x^{k-1}$ because x is always an element of M_x^k. Therefore $\hat{A}_x = A_{\phi(x)} \subseteq T_x$. $\qquad\square$

Before proving the remaining points let us fix some notation. Given $P \in [\omega_1]^{<\omega}$ and $k \in \omega$, let $A_P^k = \bigcup \{ A_\alpha^k : \alpha \in P \}$. Note that if $P, Q \in [\omega_1]^{<\omega}$ and $k > \rho^{P \cup Q}$ then, by virtue of the key feature of $\mathcal{A}$ highlighted just before this theorem, we have the following:

$$A_P^k \cap A_Q^k = A_{P \cap Q}^k,$$

$$A_P^k \cup A_Q^k = A_{P \cup Q}^k,$$

$$A_P^k \backslash A_Q^k = A_{P \backslash Q}^k.$$

We will apply these three equalities for finite sets of the form $\phi[M_x^k]$. More precisely, observe that if $x, y \in X$ and $k > \rho(\phi(x), \phi(y))$ then $\rho^{\phi[M_x^k] \cup \phi[M_y^k]} < k + 1$. This is because if we take $F \in \mathcal{F}_k$ with $\phi(x), \phi(y) \in F$, then $\phi[M_x^k] \cup \phi[M_y^k] = \phi[M_x^k \cup M_y^k] \subseteq (\phi(x))_k \cup (\phi(y))_k \subseteq F$.

Proof of (c). Let $x, y \in X$ and suppose that $\inf(x, y)$ exists. Take an arbitrary $k > \rho^{\phi[\{x,y,\inf(x,y)\}]}$. By means of the point (1) of Lemma 3.1.19, $M_x^k \cap M_y^k = M_{\inf(x,y)}^k$. In this way:

$$T_x^k \cap T_y^k = A_{M_x^k}^{k+1} \cap A_{M_y^k}^{k+1} = A_{M_x^k \cap M_y^k}^{k+1} = A_{M_{\inf(x,y)}^k}^{k+1} = T_{\inf(x,y)}^k.$$

Therefore, $T_x \cap T_y = \bigcup_{k \in \omega} \left(T_x^k \cap T_y^k \right) =^* \bigcup \{ T_{\inf(x,y)}^k \ : \ k > \rho^{\phi[x,y,\inf(x,y)]} \} =^* T_{\inf(x,y)}$. □

The remaining parts of the theorem are proved in a completely similar way. Because of this, we leave the calculations to the reader. ∎

The following result greatly extends Hausdorff's theorem about the existence of an (ω_1, ω_1)-gap.

Theorem 3.1.21. *Let $(X, <_X)$ and $(Y, <_Y)$ be two partial orders with $cof(X) = cof(Y) = \omega_1$. Then there are cofinal $X' \subseteq X$ and $Y' \subseteq Y$ for which there is an (X', Y')-gap.*

Proof. Without loss of generality we may assume that $X \cap Y = \emptyset$. By applying Proposition 3.1.17 we get $X' \subseteq X$ and $Y' \subseteq Y$ two ω_1-like cofinal subsets of X and Y' respectively. Let $Z = X' \cup Y'$ and $<_Z$ be the partial order over Z induced by X' and Y'. That is, $x <_Z y$ if and only if $x, y \in X'$ and $x <_X y$ or $x, y \in Y'$ and $x <_Y y$. It is straightforward that $(Z, <_Z)$ is an ω_1-like order. Furthermore, if $x \in X'$ and $y \in Y'$ then there is no $z \in Z$ with $z \leqslant x, y$. In virtue of Theorem 3.1.20, there is a Luzin representation of Z, namely $(\mathcal{T}, \mathcal{A})$. Due to the point (e) of Definition 3.1.15 it follows that $(\langle T_x \rangle_{x \in X'}, \langle T_y \rangle_{y \in Y'})$ forms an (X', Y')-pregap. We claim that this is in fact a gap. For this, just note that any set separating $(\langle T_x \rangle_{x \in X'}, \langle T_y \rangle_{y \in Y'})$ would also separate $(\langle A_x \rangle_{x \in X'}, \langle A_y \rangle_{y \in Y'})$ by the point (a) of Definition 3.1.15. But this is impossible since $\mathcal{A}$ is Luzin (hence inseparable) and both X' and Y' are uncountable. ∎

3.1.1 Donut-separability

In this subsection we will explore in more depth the relation between almost disjoint families and gaps. When applying the argument of Theorem 3.1.21 to the orders $X = Y = \omega_1$ we get the following result.

Theorem 3.1.22. *There is an (ω_1, ω_1)-pregap $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ such that the set*

$$\{L_{\alpha+1} \backslash L_\alpha \ : \ \alpha \in \omega_1\} \cup \{R_{\alpha+1} \backslash R_\alpha \ : \ \alpha \in \omega_1\}$$

forms a Luzin family. In particular $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is a gap.

Proof. Let $Z = \omega_1 \times 2$ and define the order over Z given by $(\alpha, i) <_Z (\beta, j)$ if and only if $\alpha < \beta$ and $i = j$. $(Z, <_z)$ is an ω_1-like order so that for any $\alpha, \beta \in \omega_1$, there is no $z \in Z$ with $z \leqslant (\alpha, 0), (\beta, 1)$. In particular, by Theorem 3.1.13 there is a Luzin representation $(\mathcal{T}, \mathcal{A})$ of Z. For any $\alpha \in \omega_1$, let $L_\alpha = T_{(\alpha,0)}$ and $R_\alpha = T_{(\alpha,1)}$. Then $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is a pregap due to the same arguments given in Theorem 3.1.21. In this

particular case, note that for any $\alpha \in \omega_1$ and $i \in 2$, the set $pred((\alpha + 1, i))$ satisfies the conditions of point (d) in Definition 3.1.15. Even more, $pred((\alpha + 1, i)) = \{(\alpha, i)\}$. In this way, $L_{\alpha+1} \backslash L_\alpha =^* A_{(\alpha,0)}$ and $R_{\alpha+1} \backslash R_\alpha =^* A_{(\alpha,1)}$. We conclude that

$$\{L_{\alpha+1} \backslash L_\alpha \,:\, \alpha \in \omega_1\} \cup \{R_{\alpha+1} \backslash R_\alpha \,:\, \alpha \in \omega_1\}$$

is an uncountable subset of $\mathcal{A}$. Therefore, it must also be a Luzin family. ∎

The core idea of Theorems 3.1.21 and 3.1.22 is the following: If $(\mathcal{L}, \mathcal{R})$ is a pregap and for each $L \in \mathcal{L}$ and $R \in \mathcal{R}$ we can find infinite sets $A_L \subseteq L$ and $A_R \subseteq R$ such that the family $\{A_L\}_{L \in \mathcal{L}} \cup \{A_R\}_{R \in \mathcal{R}}$ forms an inseparable family, then the original pregap is in fact a gap. Such phenomenom was studied in [59] by Piotr Kalemba and Szymon Plewik for the case of (ω_1, ω_1)-gaps. The main difference between the gaps they studied and the one we constructed in the previous theorem, is that the associated Luzin family can be explicitly defined from such gap. This leads to the following definition.

Definition 3.1.23 (donut-separable gaps)**.** We call an (ω_1, ω_1)-gap $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ *donut-separable* if there is a set separating $(L_{\alpha+1} \backslash L_\alpha, R_{\alpha+1} \backslash R_\alpha)_{\alpha \in \omega_1}$. If no such set exists we call the gap *donut-inseparable*.

Corollary 3.1.24. *There is a donut-inseparable Hausdorff gap.*

In contrast, there is also (in ZFC) a Hausdorff gap which is donut-separable.

Theorem 3.1.25. *The Hausdorff gap $(\mathcal{A}, \mathcal{B})$ constructed in Theorem 3.1.10 is donut-separable.*

Proof. Let $\alpha \in \omega_1$. Now consider an arbitrary $k > \rho(\alpha, \alpha + 1)$. Note that $\|\alpha\|_k + 1 = \|\alpha + 1\|_k$. By the point (c) of Lemma 2.3.9 we know that either $\Xi_{\alpha+1}(k) = \Xi_\alpha(k)$ or these two numbers are different and $\Xi_\alpha(k) = -1$. It follows that in the latter case $r_k - 1 = \|\alpha\|_k < \|\alpha + 1\|_k = r_k < m_k$, which means that $\Xi_{\alpha+1}(k) = 0$. From this fact we conclude that

$$L_{\alpha+1} \backslash L_\alpha =^* \{2k + \Xi_{\alpha+1}(k) \,:\, k > \rho(\alpha, \alpha + 1) \text{ and } \Xi_\alpha(k) \neq \Xi_{\alpha+1}(k)\}$$
$$\subseteq \{2k \,:\, k \in \omega \text{ and } \Xi_{\alpha+1}(k) = 0\} \subseteq \{2k \,:\, k \in \omega\}.$$

In the same way, we have that $R_{\alpha+1} \backslash R_\alpha \subseteq^* \{2k + 1 \,:\, k \in \omega\}$. Hence, the set of even numbers separates $(L_{\alpha+1} \backslash L_\alpha, R_{\alpha+1} \backslash R_\alpha)_{\alpha \in \omega_1}$. ∎

Lemma 3.1.26. *Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-gap and $\mathcal{L}'$, $\mathcal{R}'$ be cofinal subsets of $\mathcal{L}$ and $\mathcal{R}$ respectively. If $(\mathcal{L}, \mathcal{R})$ is donut-inseparable then so is $(\mathcal{L}', \mathcal{R}')$.*

The previous lemma motivates the following definition.

Definition 3.1.27 (Strongly donut-separable gaps)**.** Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-gap. We say that $(\mathcal{L}, \mathcal{R})$ is strongly *donut-separable* if for any two cofinal subsets of $\mathcal{L}$ and $\mathcal{R}$, say $\mathcal{L}'$ and $\mathcal{R}'$, the gap $(\mathcal{L}', \mathcal{R}')$ is donut-separable.

It is natural to wonder if there is a strongly donut-separable gap. We will prove that this statement is independent from ZFC.

Lemma 3.1.28. *Let $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap. The following statements are equivalent:*

(1) $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ *is strongly donut-separable.*

(2) *For any $S \in [\omega_1]^{\omega_1}$, the gap $(L_\alpha, R_\alpha)_{\alpha \in S}$ is donut-separable.*

(3) *For any club $S \subseteq \omega_1$, the gap $(L_\alpha, R_\alpha)_{\alpha \in S}$ is donut-separable.*

Proof. The only nontrivial part of the proof is to show that (3) implies (1). For this, let X and Y be uncountable subsets of ω_1. Our goal is to prove that there is a $C \in [\omega]^\omega$ which separates $(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{Y(\alpha+1)} \backslash R_{Y(\alpha)})_{\alpha \in \omega_1}$. Let $\langle M_\alpha \rangle_{\alpha \in \omega_1}$ be a continuous chain of elementary submodels of a largely enough $H(\lambda)$ so that $X, Y \in M_0$. Now define S as $\{0\} \cup \{M_\alpha \cap \omega_1 : \alpha \in \omega_1\}$. Then S is a club. Furthermore, by elementarity it is straightforward that for any $\alpha \in \omega_1$ there is $\beta \in S$ with:

- $S(\beta) \leqslant X(\alpha) < X(\alpha+1) \leqslant S(\beta+1)$.   - $S(\beta) \leqslant Y(\alpha) < Y(\alpha+1) \leqslant S(\beta+1)$.

By the hypotheses, there is $C \in [\omega]^\omega$ separating $(L_{S(\alpha+1)} \backslash L_{S(\alpha)}, R_{S(\alpha+1)} \backslash R_{S(\alpha)})_{\alpha \in \omega_1}$. Note that if $\alpha \in \omega_1$ and $\beta \in S$ are as previously stated then $L_{X(\alpha+1)} \backslash L_{X(\alpha)} \subseteq^* L_{S(\beta+1)} \backslash L_{S(\beta)} \subseteq^* C$ and $R_{Y(\alpha+1)} \backslash R_{Y(\alpha)} \subseteq^* R_{Y(\beta+1)} \backslash R_{Y(\beta)} \subseteq^* \omega \backslash C$. Therefore C separates the pregap $(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{Y(\alpha+1)} \backslash R_{Y(\alpha)})_{\alpha \in \omega_1}$. This finishes the proof. ∎

Corollary 3.1.29. *Let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap which is strongly donut-separable. If $\mathbb{P}$ is a ccc forcing, then*

$$\mathbb{P} \Vdash \text{`` } (\mathcal{L}, \mathcal{R}) \text{ is strongly donut-separable''.}$$

Proof. Suppose towards a contradiction that there is G a $\mathbb{P}$-generic filter over V such that, in $V[G]$, $(\mathcal{L}, \mathcal{R})$ is not strongly donut-separable. According to the point (3) of Lemma 3.1.28, there is a club $S \in V[G]$ such that the gap $(L_\alpha, R_\alpha)_{\alpha \in S}$ is donut-inseparable. Since $\mathbb{P}$ is a *ccc* forcing, we can find a club $C \in V$ for which $C \subseteq S$. is donut-inseparable and belongs to V. This contradiction ends the proof. ∎

Suppose that $(\mathcal{L}, \mathcal{R})$ is a pregap over ω and C is an infinite subset separating it. Define $s : \mathcal{L} \cup \mathcal{R} \longrightarrow \omega$ as:

$$s_C(X) = \begin{cases} \min(n : X \backslash n \subseteq C) & \text{if } X \in \mathcal{L} \\ \min(n : X \backslash n \cap C = \emptyset) & \text{if } X \in \mathcal{R} \end{cases}$$

Then $L \cap R \subseteq \max(s_C(L), s_C(R))$ for any $L \in \mathcal{L}$ and $R \in \mathcal{R}$. This motivates the following definition.

Definition 3.1.30 (Separating functions)**.** Let $(\mathcal{L}, \mathcal{R})$ be a pregap. We say that $s : \mathcal{L} \cup \mathcal{R} \longrightarrow \omega$ is *separating* if $L \cap R \subseteq \max(s(L), s(R))$ for any $L \in \mathcal{L}$ and $R \in \mathcal{R}$.

Lemma 3.1.31. *Let $(\mathcal{L}, \mathcal{R})$ be a pregap. If there is a separating $s : \mathcal{L} \cup \mathcal{R} \longrightarrow \omega$ then there is $C \in [\omega]^\omega$ which separates $(\mathcal{L}, \mathcal{R})$.*

Proof. Let s be as in the hypotheses. We define C as

$$\{n \in \omega \ : \ \exists L \in \mathcal{L} \, (n \in L \ and \ s(L) < n)\} = \bigcup_{L \in \mathcal{L}} L \backslash (s(L) + 1).$$

Note that $L \subseteq^* C$ for any $L \in \mathcal{L}$. For $R \in \mathcal{R}$ we claim that $R \backslash (s(R) + 1) \cap C = \emptyset$ which in particular implies that $R \cap C =^* \emptyset$. Suppose towards a contradiction that this is not the case and let n be an element in the intersection of both sets. On one hand, since $n \in C$ there is $L \in \mathcal{L}$ with $n \in L$ and $s(L) < n$. In particular, $n \in L \cap R$. On the other hand, since s is separating then $L \cap R \subseteq \max(s(R), s(L)) \subseteq n$. This is a contradiction. Therefore the claim is true, which means that C separates $(\mathcal{L}, \mathcal{R})$. $\blacksquare$

Definition 3.1.32. Let $(\mathcal{D}, \mathcal{E}) = (D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ be a pregap (not necessarily of type (ω_1, ω_1)) so that $D_\alpha \cap E_\alpha = \emptyset$ for each α. We define the forcing $\mathbb{P}(\mathcal{D}, \mathcal{E}) = \mathbb{P}(D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ as the set of all functions $p ; \omega_1 \longrightarrow \omega$ with finite domain and such that $D_\alpha \cap E_\beta \subseteq \max(p(\alpha), p(\beta))$ for all $\alpha, \beta \in dom(p)$. The order is given by

$$p \leqslant q \text{ if and only if } q \subseteq p.$$

Remark 3.1.33. If $(\mathcal{D}, \mathcal{E}) = (D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ is as in the previous definition and $\beta \in \omega_1$ then $\mathcal{M}_\beta = \{p \in \mathbb{P}(\mathcal{D}, \mathcal{E}) \ : \ \beta \in dom(p)\}$ is dense in $\mathbb{P}(\mathcal{E}, \mathcal{D})$. If $\mathcal{G}$ is a filter intersecting each $\mathcal{M}_\beta$ then the function $s : \mathcal{D} \cup \mathcal{E} \longrightarrow \omega_1$ given by:

$$s(D_\alpha) = s(E_\alpha) = \bigcup \mathcal{G}(\alpha)$$

is well defined and separating.

The following proposition generalizes a well-known result of Kenneth Kunen (we reiterate that our pregaps do not need to be linearly ordered).

Proposition 3.1.34. *Let $(E_\alpha, D_\alpha)_{\alpha \in \omega_1}$ be a pregap with $E_\alpha \cap D_\alpha = \emptyset$ for each α. The following statements are equivalent:*

(a) $\mathbb{P} = \mathbb{P}(E_\alpha, D_\alpha)_{\alpha \in \omega_1}$ *is ccc.*

(b) *There is $\mathbb{Q}$ ccc with $\mathbb{Q} \Vdash$ " $(E_\alpha, D_\alpha)_{\alpha \in \omega_1}$ can be separated ".*

(c) *There is W a transitive model of ZFC extending V with $\omega_1^W = \omega_1^V$ where*

$$W \models (E_\alpha, D_\alpha)_{\alpha \in \omega_1} \text{ can be separated.}$$

Proof. Trivially (a) implies (b) and (b) implies (c). In order to prove that (c) implies (a) take an arbitrary uncountable subset $\mathbb{P}$ in V, say $\mathcal{A}$. For any $p \in \mathcal{A}$ let $d_p = dom(p)$. Without loss of generality we can suppose that $\mathcal{A}$ is a Δ-system with root R, there is $n \in \omega$ so that $|p| = n$ for every $p \in \mathcal{A}$ and $p(d_p(i)) = q(d_q(i))$ for all $p, q \in \mathcal{A}$ and each $i < n$.

Let W be as in the hypotheses of (c) and let $C \in W$ which separates $(E_\alpha, D_\alpha)_{\alpha \in \omega_1}$. Since $\omega_1^V = \omega_1^W$ and V models that $\mathcal{A}$ is uncountable, we can find an uncountable $\mathcal{A}' \subseteq \mathcal{A}$ in W, so that $E_{d_p(i)} \backslash C = E_{d_q(i)} \backslash C$ and $D_{d_p(i)} \cap C = D_{d_q(i)} \cap C$ for all $p, q \in \mathcal{A}'$

and $i < n$. Fix two distinct $p, q \in \mathcal{A}'$. We claim that $p \cup q \in \mathbb{P}$. Indeed, let $i, j < n$. Then

$$
\begin{aligned}
E_{d_p(i)} \cap D_{d_q(j)} &= ((E_{d_p(i)} \backslash C) \cap D_{d_q(j)}) \cup (E_{d_p(i)} \cap (D_{d_q(j)} \cap C)) \\
&= ((E_{d_q(i)} \backslash C) \cap D_{d_q(j)}) \cup (E_{d_p(i)} \cap (D_{d_p(j)} \cap C)) \\
&\subseteq \max(q(d_q(i)), q(d_q(j)) \cup \max(p(d_p(i)), p(d_p(j))) \\
&= \max(p(d_p(i)), p(d_q(j))).
\end{aligned}
$$

This finishes the proof. ∎

Consider the gap $(L_\alpha, R_\alpha)_{\alpha \in \omega}$ constructed in Theorem 3.1.10 and for any $k \in \omega$ let $N_k = 2(k+1) = \{0, \dots, 2k+1\}$. The following properties follow directly from the proof of that theorem and the definition of the Δ-function.

Proposition 3.1.35. *Let $\alpha < \beta \in \omega_1$. Then:*

(0) $L_\alpha \cap R_\alpha = \emptyset$.

(1) *For each $k \in \omega$, both $\{2k, 2k+1\} \cap L_\alpha$ and $\{2k, 2k+1\} \cap R_\alpha$ have at most one point.*

(2) *All of the sets $L_\alpha \backslash L_\beta$, $R_\alpha \backslash R_\beta$, $L_\alpha \cap R_\beta$ and $L_\beta \cap R_\alpha$ are subsets of $N_{\rho(\alpha,\beta)}$.*

(3) $L_\alpha \cap N_{\Delta(\alpha,\beta)-1} = L_\beta \cap N_{\Delta(\alpha,\beta)-1}$ *and* $R_\alpha \cap N_{\Delta(\alpha,\beta)-1} = R_\beta \cap N_{\Delta(\alpha,\beta)-1}$.

Let us briefly recall Definition 2.4.17: If $\mathcal{F}$ is a 2-capturing construction scheme over ω_1, the parametrized Martin's number $\mathfrak{m}_\mathcal{F}$ associated to $\mathcal{F}$, is the minimun of the Martin's numbers of *ccc* forcing notions which force $\mathcal{F}$ to be 2-capturing in the generic extension.

In the following theorem, we will show that the existence of (ω_1, ω_1)-gaps which are strongly donut-separable is consistent with ZFC. The proof we provide can be simplified by the use of Lemma 4.0.3. We decided to keep it this way in order to motivate such lemma.

Theorem 3.1.36. *Let $\mathcal{F}$ be a 2-capturing 2-construction scheme for which $\mathfrak{m}_\mathcal{F} > \omega_1$. The Hausdorff gap constructed from $\mathcal{F}$ in Theorem 3.1.10 is strongly donut-separable.*

Proof. Let $S \in [\omega_1]^{\omega_1}$. We will show that the gap $(L_{S(\alpha)}, R_{S(\alpha)})_{\alpha \in \omega_1}$ is donut-separable. In other words, we will show that pregap $(D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ can be separated where $D_\alpha = L_{S(\alpha+1)} \backslash L_{S(\alpha)}$ and $E_\alpha = R_{S(\alpha+1)} \backslash R_{S(\alpha)}$ for each $\alpha \in \omega_1$. This is enough due to the Lemma 3.1.28. In view of Remark 3.1.33 and since we are assuming that $\mathfrak{m}_\mathcal{F} > \omega_1$, it is sufficient to prove that for $\mathbb{P} = \mathbb{P}(D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ we have:

(1) $\mathbb{P}$ is a *ccc* forcing,

(2) $\mathbb{P} \Vdash$ " $\mathcal{F}$ *is 2-capturing* ".

Proof of (1). For this consider an arbitrary $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$. Given $p \in \mathcal{A}$ define $Z_p = \{S(\alpha), S(\alpha+1) : \alpha \in dom(p)\}$. By refining $\mathcal{A}$ we can assume without loss of generality that for any $p, q \in \mathcal{A}$ the following conditions hold:

(a) $|p| = |q|$,

(b) If $f : dom(p) \longrightarrow dom(q)$ is the increasing bijection then $p(\alpha) = q(f(\alpha))$ for each $\alpha \in dom(p)$. In particular, there is $k \in \omega$ with $k > \max(im(p))$ for each $p \in \mathcal{A}$.

Furthermore, we can suppose that the set $\{dom(p) : p \in \mathcal{A}\}$ forms a root-tail-tail Δ-system with root R satisfying the following properties for any two distinct $p, q \in \mathcal{A}$:

(c) If $R \neq \emptyset$, then $\max(R) + 1 < \min(dom(p)\backslash R)$.

(d) $\max(dom(p)) + 1 \notin dom(q)$.

As the set $\{Z_p : p \in \mathcal{A}\}$ is uncountable and $\mathcal{F}$ is assumed to be 2-capturing then there are distinct $p, q \in \mathcal{A}$ for the set $\{Z_p, Z_q\}$ is captured at some level $l > k$. We affirm that p and q are compatible. This will follow from the next claim.

<u>Claim 1:</u> $p \cup q$ is a condition of $\mathbb{P}$.

Proof of claim. First note that if $F \in \mathcal{F}_l$ is such that $Z_p \cup Z_q \subseteq F$ then $Z_p \cap Z_q = Z_p \cap R(F) = Z_q \cap R(F)$ by means of Lemma 2.4.5.

Now, let us consider $f : dom(p) \longrightarrow dom(q)$ and $h : F_0 \longrightarrow F_1$ the increasing bijections. In order to prove that $p \cup q$ is a condition it is enough to take $\alpha \in dom(p)$ and $\beta \in dom(q)$ and show that both $E_\alpha \cap D_\beta$ and $D_\alpha \cap E_\beta$ are contained in $\max(p(\alpha), q(\beta))$. If either α or β belong to the intersection of $dom(p) \cap dom(q)$ there is nothing to do. So we can assume that $\alpha \in dom(p)\backslash dom(q)$ and $\beta \in dom(q)\backslash dom(p)$. Then $\{S(\alpha), S(\alpha + 1)\} \subseteq Z_p\backslash Z_q \subseteq F_0\backslash R(F)$ and $\{S(\beta), S(\beta + 1)\} \subseteq Z_q\backslash Z_p \subseteq F_1\backslash R(F)$ due to the points (c) and (d). Thus $\rho(S(\alpha + 1), S(\beta + 1)) = l$ and consequently $D_\alpha \cap E_\beta \subseteq L_{S(\alpha+1)} \cap R_{S(\beta+1)} \subseteq N_l$ due to the point (2) of Proposition 3.1.35. But $\Xi_{S(\alpha)}(l) = \Xi_{S(\alpha+1)}(l) = 0$. Therefore $2l \in L_{S(\alpha)} \cap L_{S(\alpha+1)}$. In particular $2l \in L_{S(\alpha+1)}$, so $2l+1 \notin L_{S(\alpha+1)}$ by the point (1) of Proposition 3.1.35. In this way $\{2l, 2l+1\} \cap D_\alpha = \emptyset$. Thus,

$$D_\alpha \cap E_\beta \subseteq N_{l-1}.$$

The next thing to note is that $h(S(\alpha)) = S(f(\alpha))$ and $h(S(\alpha+1)) = S(f(\alpha)+1)$. This means that $\Delta(S(\alpha), S(f(\alpha))) = l = \Delta(S(\alpha + 1), S(f(\alpha) + 1))$. In virtue of the point (3) of proposition 3.1.35 we have

$$L_{S(\alpha)} \cap N_{l-1} = L_{S(f(\alpha))} \cap N_{l-1},$$

$$L_{S(\alpha+1)} \cap N_{l-1} = L_{S(f(\alpha)+1)} \cap N_{l-1}.$$

Hence, $D_\alpha \cap N_{l-1} = D_{f(\alpha)} \cap N_{l-1}$. From all the equations we have so far we deduce that $D_\alpha \cap E_\beta = D_{f(\alpha)} \cap E_\beta \subseteq \max(q(f(\alpha)), q(\beta)) = \max(p(\alpha), q(\beta))$. In a completely similar way we can show that $E_\alpha \cap D_\beta \subseteq \max(p(\alpha), q(\beta))$. We conclude that $p \cup q \in \mathbb{P}$ so we are done. $\square$

$\square$

Proof of (2). For this we will show that the equivalence of 2-capturing stated in Lemma 2.4.15 is forced by $\mathbb{P}$. Let $\dot{X}$ be a name so that $\mathbb{1}_{\mathbb{P}} \Vdash$ " $\dot{X} \in [\omega_1]^{\omega_1}$" and let $q \in \mathbb{P}$. We shall find $p \leqslant q$ so that

$$p \Vdash \text{ “ } \exists M \in [\dot{X}]^2 \, (\, M \text{ is captured }) \text{ ”}.$$

As $\dot{X}$ is forced to be uncountable we can find for each $\alpha \in \omega_1$ a condition $p_\alpha \leqslant q$ and $\xi_\alpha > \alpha$ with $p_\alpha \Vdash$ " $\xi_\alpha \in \dot{X}$ ". We can suppose without loss of generality that the family $\mathcal{A} = \{p_\alpha : \alpha \in \omega_1\}$ satisfies the conditions (a), (b), (c) and (d) of the previous paragraphs. We may also assume that for any distinct $\alpha, \beta \in \omega_1$:

(d) $\xi_\alpha \neq \xi_\beta$ and $|Z_{p_\alpha} \cup \{\xi_\alpha\}| = |Z_{p_\beta} \cup \{\xi_\beta\}|$,

(e) If $g : Z_{p_\alpha} \cup \{\xi_\alpha\} \longrightarrow Z_{p_\beta} \cup \{\xi_\beta\}$ is the increasing bijection then $g[Z_{p_\alpha}] = Z_{p_\beta}$ and $g(\xi_\alpha) = \xi_\beta$.

Since $\mathcal{F}$ is 2-capturing there are $\alpha < \beta \in \omega_1$ for which $\{Z_{p_\alpha} \cup \{\xi_\alpha\}, Z_{p_\beta} \cup \{\xi_\beta\}\}$ is captured at some level l greater than k. In virtue of the conditions (d) and (e) it is easy to see that $\{Z_{p_\alpha}, Z_{p_\beta}\}$ and $\{\xi_\alpha, \xi_\beta\}$ are also captured at level l. As in the first part of the proof, this implies that $p = p_\alpha \cup p_\beta$ is a condition of $\mathbb{P}$. To finish, just note that $p \leqslant q$ and $p \Vdash$ " $\{\xi_\alpha, \xi_\beta\} \in [\dot{X}]^2$ *and* $\{\xi_\alpha, \xi_\beta\}$ *is captured* ". This finishes the proof. $\square$

$\blacksquare$

Definition 3.1.37 (Adequate sets). Let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap. We say that $X \in [\omega_1]^{\omega_1}$ is *adequate* for $(\mathcal{L}, \mathcal{R})$ if for any $Y \in [\omega_1]^{\omega_1}$, the pregap

$$(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{X(\alpha+1)} \backslash R_{X(\alpha)})_{\alpha \in Y}$$

is a gap.

Remark 3.1.38. If an (ω_1, ω_1)-pregap admits an adequate set, then such pregap is not strongly donut-separable.

Theorem 3.1.39 (Under CH). *Let* $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ *be an* (ω_1, ω_1)-gap. *Then there is a club* $X \in [\omega_1]^{\omega_1}$ *adequate for* $(\mathcal{L}, \mathcal{R})$. *In particular, there are no strongly-donut separable gaps.*

Proof. Since we are assuming CH we can enumerate $[\omega]^\omega$ as $\langle C_\alpha \rangle_{\alpha \in \omega_1}$. We will build X be recursion in such way that for any $\beta \in \omega_1$ and each $\alpha \leqslant \beta$ one of the following conditions occur:

$$\textbf{(A)} \qquad\qquad\qquad\qquad \textbf{(B)}$$

$$L_{X(\beta+1)} \backslash L_{X(\beta)} \not\subseteq^* C_\alpha. \qquad\qquad (R_{X(\beta+1)} \backslash R_{X(\beta)}) \cap C_\alpha \neq^* \emptyset.$$

If γ is limit and we have constructed $X(\alpha)$ for any $\alpha < \gamma$ we just define $X(\gamma)$ as $\sup(X(\alpha) : \alpha < \gamma)$. The interesting case happens when we have constucted $X(\beta)$ for some $\beta \in \omega_1$ and we want to define $X(\beta+1)$. For this case, we have the following claim:

Claim: The pregap $(L_\delta \backslash L_{X(\beta)}, R_\delta \backslash R_{X(\beta)})_{\delta > X(\beta)}$ is a gap.

Proof of claim. Suppose towards a contradiction that this is not the case and let C be a set separating $(L_\delta \backslash L_{X(\beta)}, R_\delta \backslash R_{X(\beta)})_{\delta > X(\beta)}$. Then $C \cup L_{X(\beta)}$ separates $(\mathcal{L}, \mathcal{R})$ which is a contradiction. $\square$

Fix $\alpha \leqslant \beta$. By the previous claim, C_α does not separate $(L_\delta \backslash L_{X(\beta)}, R_\delta \backslash R_{X(\beta)})_{\delta > X(\beta)}$. In this way, there is $\delta_\alpha > X(\beta)$ so that either $L_{\delta_\alpha} \backslash L_{X(\beta)} \not\subseteq^* C_\alpha$ or $R_{\delta_\alpha} \cap C_\alpha \neq^* \emptyset$. Let us define $X(\beta + 1)$ as $\sup(\delta_\alpha : \alpha \leqslant \beta)$. Then $L_{\delta_\alpha} \backslash L_{X(\beta)} \subseteq L_{X(\beta+1)} \backslash L_{X(\beta)}$ and $R_{\delta_\alpha} \backslash R_{X(\beta)} \subseteq R_{X(\beta+1)} \backslash R_{X(\beta)}$ for each $\alpha \leqslant \beta$. In this way we guarantee that either contition (A) or condition (B) will ocurr for any such α. This finishes the recursion.

We will now prove that X is as desired. Let $Y \in [\omega_1]^{\omega_1}$. We need to show that $(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{X(\alpha+1)} \backslash R_{X(\alpha)})_{\alpha \in Y}$ is a gap. For this let C be an infinite subset of ω. By the assumptions we know there is $\alpha \in \omega_1$ so that $C = C_\alpha$. Let $\beta \in Y$ be such that $\beta > \alpha$. Therefore, either $L_{X(\beta+1)} \backslash L_{X(\beta)} \not\subseteq^* C_\alpha$ or $(R_{X(\beta+1)} \backslash R_{X(\beta)}) \cap C_\alpha \neq^* \emptyset$. In any case, $X(\beta)$ testifies that C_α does not separate the gap that we are considering. This finishes the proof. $\blacksquare$

By combining the previous result and Theorem 3.1.36, we conclude the following:

Theorem 3.1.40. *The statement " There is a strongly donut-separable gap" is independent from ZFC.*

Now we will analyze the relationship between donut-separable gaps and the Cohen forcing.

Lemma 3.1.41. *Let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap and let $X \in [\omega_1]^{\omega_1}$. If X is adequate for $(\mathcal{L}, \mathcal{R})$ then $\mathbb{C} \Vdash$ " X is adequate for $(\mathcal{L}, \mathcal{R})$".*

Proof. Let G be a $\mathbb{C}$-generic filter over V and let $Y \in [\omega_1]^{\omega_1} \cap V[G]$. Suppose towards a contradiction that there is $C \in V[G]$ separating the pregap

$$(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{X(\alpha+1)} \backslash R_{X(\alpha)})_{\alpha \in Y}.$$

Since Y is uncountable we can find $a, b \in [\omega]^{<\omega}$ and $Y' \in [Y]^{\omega_1} \cap V[G]$ so that for any $\alpha \in Y'$, $(L_{X(\alpha+1)} \backslash L_{X(\alpha)}) \backslash C = a$. Due to well-known facts concerning the Cohen forcing $\mathbb{C}$, we know there is $Z \in [\omega_1]^{\omega_1} \cap V$ with $Z \subseteq Y$. Let us define

$$C' = \bigcup_{\alpha \in Z} L_{X(\alpha+1)} \backslash L_{X(\alpha)}.$$

Then $C' \in V$ and trivially $L_{X(\alpha+1)} \backslash L_{X(\alpha)} \subseteq C'$ for any $\alpha \in Z$. Furthermore, $C' \subseteq C \cup a$. Therefore, $(R_{X(\alpha+1)} \backslash R_{X(\alpha)}) \cap C'$ is finite for any $\alpha \in Z$. We conclude that, in V, there is a set separating the gap $(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{X(\alpha+1)} \backslash R_{X(\alpha)})_{\alpha \in Z}$. This is a contradiction to the hypotheses. Hence, the proof is over. $\blacksquare$

Theorem 3.1.42 (Under CH). *Let κ be an uncountable cardinal. Then*

$$\mathbb{C}_\kappa \Vdash \text{ " There are no strongly donut-separable } (\omega_1, \omega_1)\text{-gaps "}.$$

In particular, the statement "There are no strongly donut-separable (ω_1, ω_1)-gaps" is consistent with an arbitrarily large continuum.

Proof. If $\kappa = \omega_1$ the argument is clear, since in this case $\mathbb{C}_\kappa$ forces CH. So suppose that $\kappa > \omega_1$ and let G be a $\mathbb{C}_\kappa$-generic filter over V. Finally, let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1} \in V[G]$ be an arbitrary (ω_1, ω_1)-gap. Since $|\mathcal{L}| = |\mathcal{R}| = \omega_1$, then there is a $\mathbb{C}_{\omega_1}$-generic filter over V, namely H, and a $\mathbb{C}_\kappa$-generic filter over $V[H]$, namely K, so that $V[H][K] = V[G]$ and $(\mathcal{L}, \mathcal{R}) \in V[H]$. By the hypotheses, V is a model of CH. Therefore $V[H]$ models CH too. According to Theorem 3.1.39 there is $X \in V[H]$ which is adequate for $(\mathcal{L}, \mathcal{R})$. We finish by proving the following claim.

Claim: X testifies that $(\mathcal{L}, \mathcal{R})$ is not strongly donut-separable.

Proof of claim. Suppose towards a contradiction that there is $C \in V[G]$ separating the pregap

$$(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{X(\alpha+1)} \backslash R_{X(\alpha)})_{\alpha \in \omega_1}.$$

Again, since $|C| = \omega$ we can find $\mathbb{C}$-generic filter over $V[H]$, say H', for which $C \in V[H][H']$. By virtue of Lemma 3.1.41, $V[H][H'] \models X$ *is adequate for* $(\mathcal{L}, \mathcal{R})$. In particular C can not separate the pregap that we are considering. This contradiction finishes the proof. $\qquad \square$

$\blacksquare$

As $\mathfrak{m}_\mathcal{F} > \omega_1$ is consistent with $\mathfrak{c}$ arbitrarily large, we conclude that both the statement "there is a strongly donut-separable gap" and its negation are consistent with an arbitrarily large continuum.

Our next goal is to show that PFA also implies that there are no strongly donut-separable gaps. For this sake, we need to consider the following definitions.

Definition 3.1.43 (Almost Luzin gaps)**.** Let $(\mathcal{D}, \mathcal{E}) = (D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ be pregap (not necessarily of type (ω_1, ω_1)). We say that $(\mathcal{D}, \mathcal{E})$ is *almost Luzin* if $(\mathcal{D}', \mathcal{E}')$ is a gap for any two uncountable $\mathcal{D}' \subseteq \mathcal{D}$ and $\mathcal{E}' \subseteq \mathcal{E}$.

Definition 3.1.44 (Highly adequate sets)**.** Let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap. We say that $X \in [\omega_1]^{\omega_1}$ is *highly adequate* for $(\mathcal{L}, \mathcal{R})$ if the pregap

$$(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{X(\alpha+1)} \backslash R_{(X(\alpha)})_{\alpha \in X}$$

is almost Luzin.

Remark 1. If X is highly adequate for $(\mathcal{L}, \mathcal{R})$ then it is also adequate.

It is very easy check that the dichotomy satisfied by the club X that we constructed in Theorem 3.1.39 already turns X into a highly adequate set for the gap $(\mathcal{L}, \mathcal{R})$. Hence, we have the following proposition.

Proposition 3.1.45 (Under CH)**.** *Let* $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ *be an* (ω_1, ω_1)-*gap. Then there is a club* $X \in [\omega_1]^{\omega_1}$ *highly adequate for* $(\mathcal{L}, \mathcal{R})$.

Definition 3.1.46 (Normal pregaps)**.** Suppose that $(D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ is a pregap. We say that it is *normal* if $D_\alpha \cap E_\alpha = \emptyset$ for each $\alpha \in \omega_1$.

Remark 3.1.47. Suppose that $(D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ is a pregap and consider, for each α, $D'_\alpha = D_\alpha \backslash E_\alpha$ and $E'_\alpha = E_\alpha \backslash D_\alpha$. Then $(D'_\alpha, E'_\alpha)_{\alpha \in \omega_1}$ is a normal pregap. Furthermore, both gaps induce the same sets in $\mathscr{P}(\omega)/\text{FIN}$. In particular, $(D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ is strongly donut-separable if and only if the same holds for $(D'_\alpha, E'_\alpha)_{\alpha \in \omega_1}$.

Definition 3.1.48 (Biorthogonal gaps)**.** Let $(\mathcal{D}, \mathcal{E}) = (D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ be a normal pregap. We say that $(\mathcal{D}, \mathcal{E})$ is *biorthogonal* if $(D_\alpha \cap E_\beta) \cup (D_\beta \cap E_\alpha) \neq \emptyset$ for all $\alpha \neq \beta \in \omega_1$.

Lemma 3.1.49. *If $(\mathcal{D}, \mathcal{E}) = (E_\alpha, D_\alpha)_{\alpha \in \omega_1}$ is a normal biorthogonal pregap, then it is a gap.*

Proof. Suppose towards a contradiction that there is $C \in [\omega]^\omega$ which separates $(\mathcal{D}, \mathcal{E})$. According to the pigheonhole principle, there is $X \in [\omega_1]^{\omega_1}$ for which $D_\alpha \backslash C = D_\beta \backslash C$ and $E_\alpha \cap C = E_\beta \cap C$ for all $\alpha, \beta \in X$. Fix two distinct $\alpha, \beta \in X$. Then

$$D_\alpha \cap E_\beta = (D_\alpha \cap (E_\beta \cap C)) \cup (E_\beta \cap (D_\alpha \backslash C))$$
$$= (D_\alpha \cap (E_\alpha \cap C)) \cup (E_\beta \cap (D_\beta \backslash C)) = \emptyset$$

By symmetry, we also have that $D_\beta \cap E_\alpha = \emptyset$, but this is a contradiction to the normality of $(\mathcal{D}, \mathcal{E})$. Thus, the proof is over. ∎

Definition 3.1.50. Let $(\mathcal{D}, \mathcal{E}) = (D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ be a normal gap. We define the forcing $\mathbb{L}(\mathcal{D}, \mathcal{E})$ as the set of all $p \in [\omega_1]^{<\omega}$ so that $(D_\alpha \cap E_\beta) \cup (D_\beta \cap E_\alpha) \neq \emptyset$ for all $\alpha \neq \beta \in dom(p)$. The order is given by

$$p \leqslant q \text{ if and only if } q \subseteq p.$$

Proposition 3.1.51. *Let $(\mathcal{D}, \mathcal{E}) = (D_\alpha, E_\alpha)_{\alpha \in \omega_1}$ be a normal gap. If $(\mathcal{D}, \mathcal{E})$ is almost Luzin, then $\mathbb{L}(\mathcal{D}, \mathcal{E})$ is ccc.*

Proof. Let $\mathcal{A}$ be an uncountable subset of $\mathbb{L}(\mathcal{D}, \mathcal{E})$. We will show that $\mathcal{A}$ is not an antichain. By refining $\mathcal{A}$, we may assume without loss of generality that there are $n, m \in \omega$ for which the following properties hold for each $p, q \in \mathcal{A}$:

(1) $|p| = n$.

(2) For all $i, j < n$, $(D_{p(i)} \cap E_{p(j)}) \cup (D_{p(j)} \cap E_{p(i)}) \subseteq m$.

(3) For all $i < n$, $D_{p(i)} \cap m = D_{q(i)} \cap m$ and $D_{p(i)} \cap m = D_{q(i)} \cap m$.

Note that for any two given conditions in $p, q \in \mathbb{L}(\mathcal{D}, \mathcal{E})$, we have that p and q are compatible if and only if $p \backslash q$ and $q \backslash p$ are compatible conditions. Because of this and due to the Δ-system Lemma, we may also assume that the elements of $\mathcal{A}$ are pairwise disjoint. Now, we proceed to find distinct $p, q \in \mathcal{A}$ for which $p \cup q$ is a condition. This will be done after proving the following claims.

<u>Claim 1</u>: Let $p, q \in \mathcal{A}$ and $i \neq j < n$, then $(D_{p(i)} \cap E_{q(j)}) \cup (D_{q(j)} \cap E_{p(i)}) \neq \emptyset$.

Proof of claim. Just note that by the conditions (2) and (3), we have the following chain of equalities:

$$
((D_{p(i)} \cap E_{q(j)}) \cup (D_{q(j)} \cap E_{p(i)})) \cap m = (D_{p(i)} \cap (E_{q(j)} \cap m)) \cup ((D_{q(j)} \cap m) \cap E_{p(i)})
$$
$$
= (D_{p(i)} \cap (E_{p(j)} \cap m)) \cup ((D_{p(j)} \cap m) \cap E_{p(i)})
$$
$$
= (D_{p(i)} \cap E_{p(j)}) \cup (D_{p(j)} \cap E_{p(i)}) \neq \emptyset.
$$

This proves the claim. $\square$

<u>Claim 2</u>: Let $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ and $i < n$. Then there is $\mathcal{X} \in [\mathcal{A}']^{\omega_1}$ so that for any $k \in \omega$, if $\{p \in \mathcal{X} : k \in D_{p(i)}\}$ is non-empty, then it is uncountable. Analogously with the set $\{p \in \mathcal{X} : k \in E_{p(i)}\}$.

Proof of claim. Let M be a countable elementary submodel of $H(\omega_2)$ such that $\mathcal{A}' \in M$. We put $\mathcal{X} = \mathcal{A}' \backslash M$. Note that if $k \in \omega$ is such that $k \in D_{q(i)}$ for some $q \in \mathcal{X}$, then the set $\{p \in \mathcal{A}' : k \in D_{p(i)}\}$ is an element of M which is not contained in it. By elementarity, it follows that this set is uncountable. Therefore $\{p \in \mathcal{X} : k \in D_{p(i)}\} = \{p \in \mathcal{A}' : k \in D_{p(i)}\} \backslash M$ is uncountable as well. The same argument holds for the set $\{p \in \mathcal{X} : k \in E_{p(i)}\}$. Hence, the proof is over. $\square$

<u>Claim 3</u>: Let $i < n$ and $\mathcal{X}$, $\mathcal{Y}$ be uncountable disjoint subsets of $\mathcal{A}$ satisfying the conclusions of the Claim 2 when applied to i. Then there are $\mathcal{X}' \in [\mathcal{X}]^{\omega_1}$ and $\mathcal{Y}' \in [\mathcal{X}]^{\omega_1}$ so that

$$
(D_{p(i)} \cap E_{q(i)}) \cup (D_{q(i)} \cap E_{q(i)}) \neq \emptyset
$$

for all $p \in \mathcal{X}'$ and $q \in \mathcal{Y}'$.

Proof of claim. By the hypotheses of the proposition, we have that $(\langle D_{p(i)} \rangle_{p \in \mathcal{X}}, \langle E_{q(i)} \rangle_{q \in \mathcal{Y}})$ is a gap. In particular, there is $k \in \omega$, $p' \in \mathcal{X}$ and $q' \in \mathcal{Y}$ for which $k \in D_{p'(i)} \cap E_{q'(i)}) \cup (D_{q'(i)} \cap E_{q'(i)})$. Without loss of generality we may assume that $k \in D_{p'(i)} \cap E_{q'(i)}$. We define

$$
\mathcal{X}' = \{p \in \mathcal{X} : , k \in D_{p(i)}\},
$$
$$
\mathcal{Y}' = \{q \in \mathcal{Y} : k \in E_{q(i)}\}.
$$

It is straightforward that $\mathcal{X}'$ and $\mathcal{Y}'$ are the sets that we are looking for. $\square$

By applying multiple times the claims 2 and 3, we may build two sequences $\mathcal{X}_{n-1} \subseteq \cdots \subseteq \mathcal{X}_0 \subseteq \mathcal{A}$ and $\mathcal{Y}_{n-1} \subseteq \cdots \subseteq \mathcal{Y}_0 \subseteq \mathcal{A}$ of uncountable sets for which $\mathcal{X} \cap \mathcal{Y} = \emptyset$ and such that $(D_{p(i)} \cap E_{q(i)}) \cup (D_{q(i)} \cap E_{q(i)}) \neq \emptyset$ for all $i < n$, $p \in \mathcal{X}_i$ and $q \in \mathcal{Y}_i$. Using Claim 1, it should be clear that if $p \in \mathcal{X}_{n-1}$ and $q \in \mathcal{Y}_{n-1}$, then $p \cup q$ is a condition of $\mathbb{L}(\mathcal{D}, \mathcal{E})$. Thus, we are done.

$\blacksquare$

Corollary 3.1.52. *Let $(\mathcal{D}, \mathcal{E})$ be an almost Luzin gap. Then there is a ccc forcing $\mathbb{Q}$ so that $\mathbb{Q} \Vdash$ " $(\mathcal{D}, \mathcal{E})$ has a biorthogonal subgap."*

Theorem 3.1.53 (Under PFA)**.** *There are no strongly donut-separable (ω_1, ω_1)-gaps*

Proof. Let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap and consider the forcing $\mathbb{P} = 2^{<\omega_1}$. According to the Remark 3.1.47, we may assume that $(\mathcal{L}, \mathcal{R})$ is normal. Now, let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V. Then $V[G]$ models CH. Furthermore, $[\omega]^\omega \cap V = [\omega]^\omega \cap V[G]$. In this way, $(\mathcal{L}, \mathcal{R})$ is still a gap in $V[G]$. According to the Proposition 3.1.45, we can find a club $X \subseteq \omega_1$ in $V[G]$ which is highly adequate for $(\mathcal{L}, \mathcal{R})$. That is, in $V[G]$, the gap $(L_{X(\alpha+1)} \backslash L_{X(\alpha)}, R_{X(\alpha+1)} \backslash R_{X(\alpha)})_{\alpha \in \omega_1}$ is almost Luzin. Therefore, by virtue of the Corollary 3.1.52 there is a *ccc* forcing $\mathbb{Q}$ which forces this gap to have a biorthogonal subgap. Let H be a $\mathbb{Q}$-generic filter over $V[G]$. Then, in $V[G][H]$, there is $Y \in [\omega_1]^{\omega_1}$ so that the pregap

$$(\langle L_{X(\alpha+1)} \backslash L_{X(\alpha)} \rangle, \langle R_{X(\alpha+1)} \backslash R_{X(\alpha)} \rangle)_{\alpha \in Y}$$

is biorthogonal.

Returning to V, let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$ name for $\mathbb{Q}$ in $V[G]$. Now, let $\dot{X}$ and $\dot{Y}$ be $\mathbb{P} * \dot{\mathbb{Q}}$-names for X and Y respectively. Finally, let $p \in \mathbb{P} * \dot{\mathbb{Q}}$ which forces $\dot{X}$ and $\dot{Y}$ to have the properties discussed in the previous paragraph. Given $\xi \in \omega_1$, let

$$D_\xi = \{q \leqslant p \ : \exists \xi < \alpha, \beta \in \omega_1 \, (q \Vdash \text{`` } \alpha \in \dot{X}, \text{ and } \beta \in \dot{Y}\text{)''} \},$$

$$E_\xi = \{q \leqslant p \ : \ p \Vdash \text{``} \xi \notin \dot{X}\text{''} \text{ or there is } \alpha \in \omega_1 \text{ such that } p \Vdash \text{``} \dot{X}(\alpha) = \xi\text{''}\}.$$

It is straightforward that D_ξ and E_ξ are dense below p for any $\xi \in \omega_1$. Since $\mathbb{P}$ is σ-closed and $\dot{\mathbb{Q}}$ is forced to be *ccc*, then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper. In this way, there is a filter F which intersects each D_ξ and E_ξ. Let

$$X' = \{\alpha \ : \ \exists q \in F \, (q \Vdash \text{``}\alpha \in \dot{X}\text{''})\},$$

$$Y' = \{\alpha \ : \ \exists q \in F \, (q \Vdash \text{``}\alpha \in \dot{Y}\text{''})\}.$$

Note that the statement

$$\text{``} (L_{X'(\alpha+1)} \backslash L_{X'(\alpha)} \cap R_{X'(\beta+1)} \backslash R_{X'(\beta)}) \cup (R_{X'(\alpha+1)} \backslash R_{X'(\alpha)} \cap L_{X'(\beta+1)} \backslash L_{X'(\beta)}) = \emptyset\text{''}$$

is absolute. From this, it easily follows that that the pregap

$$(L_{X'(\alpha+1)} \backslash L_{X'(\alpha)}, R_{X'(\alpha+1)} \backslash R_{X'(\alpha)})_{\alpha \in Y'}$$

is biorthogonal. In particular, it is a gap by Lemma 3.1.49. Form this, we easily have that $(\mathcal{L}, \mathcal{R})$ is not strongly donut-separable.

∎

So far we have seen that both CH and PFA imply that the non-existence of strongly donut-separable gaps. On the other hand, the existence of a 2-capturing construction scheme $\mathcal{F}$ for which $\mathfrak{m}_\mathcal{F} > \omega_1$ implies the existence of a strongly donut-separable gap. We will end this subsection by proving that MA is not sufficient to decide this problem. This result suggests that the cardinal invariant $\mathfrak{m}_\mathcal{F}$ should not be treated just as a sort of weakening of MA, but rather as an interesting principle on its own.

Theorem 3.1.54. *MA is independent from the statement "there are strongly donut-inseparable gaps".*

Proof. Since PFA implies that there are no strongly donut-inseparable gaps, then MA is consistent with that same statement. In order to show the consistency of the negation, we start with a model V in which there is a strongly donut-inseparable gap. We now consider a *ccc* forcing $\mathbb{P}$ which forces MA and G a $\mathbb{P}$-generic filter over V. Then $V[G]$ is a model MA. Furthermore, by means of the Corollary 3.1.29, $V[G]$ also models the existence of a strongly donut-inseparable gap. Thus, the proof is over.

∎

3.1.2 The gap cohomology group

In the subsection we naturally extend the work done by Daniel Talayco in [105]. In particular, we study the size of groups defined in a similar context as Talayco and prove that, under certain assumptions, these groups are as big as possible.

Definition 3.1.55 (∗-lower semi-lattice)**.** Let X be an infinite set. We say that a family $\mathcal{T} \subseteq [X]^\omega$ is a *∗-lower semi-lattice* if for any $A, B \in \mathcal{T}$ there is $C \in \mathcal{T}$ so that $A \cap B =^* C$. In other words, $\{[A] : A \in \mathcal{T}\}$ is a lower semi-lattice in $\mathscr{P}(X)/\text{FIN}$.

Remark 2. Towers are particular cases of ∗-lower semi-lattices.

Definition 3.1.56 (Coherent Subsystems)**.** Let $\mathcal{T}$ be a ∗-lower semi-lattice over a countable set X. A function $g : \mathcal{T} \longrightarrow \mathscr{P}(X)$ is said to be a *coherent subsystem of $\mathcal{T}$* if for any $A, B \in \mathcal{T}$, the following happens:

(1) $g(A) \subseteq A$,

(2) if $B \subseteq^* A$ then $g(A) \cap B =^* g(B)$.

We say that g is *trivial* if there is $C \in \mathscr{P}(X)$ so that $C \cap A =^* g(A)$ for any $A \in \mathcal{T}$. In this case, we say that C *trivializes g*. The set of all coherent subsystems of $\mathcal{T}$ is denoted as $C(\mathcal{T})$ and the set of all trivial coherent subsystem is denoted as $Tr(\mathcal{T})$.

Definition 3.1.57. Let $\mathcal{T}$ be a ∗-lower semilattice over a countable set X and g be a coherent subsystem of $\mathcal{T}$. We denote the sequence $(g(A), A \backslash g(A))_{A \in \mathcal{T}}$ as $\mathcal{C}(\mathcal{T}, g)$.

The following lemma is easy. We prove it just to emphasise the reason of defining coherent subsystems just for ∗-lower semi-lattices.

Lemma 3.1.58. *Let $\mathcal{T}$ be a ∗-lower semi-lattice and let g be a coherent subsystem of $\mathcal{T}$. Then $\mathcal{C}(\mathcal{T}, g)$ is a pregap. Furthermore, g is non-trivial if and only if $\mathcal{C}(\mathcal{T}, g)$ forms a gap.*

Proof. Let $A, B \in \mathcal{T}$. We need to prove that $g(A) \cap (B \backslash g(B)) =^* \emptyset$. For this let $C \in \mathcal{T}$ be so that $C =^* A \cap B$. Then

$$g(A) \cap B =^* (g(A) \cap A) \cap B =^* g(A) \cap C =^* g(C) =^* g(B) \cap C \subseteq^* g(B).$$

This finishes the argument. Now we will prove the proposed equivalence.

Proof of $\Rightarrow$. We will prove this by contra-positive. Suppose that there is a set C separating the pregap $\mathcal{C}(\mathcal{T}, g)$. For any $A \in \mathcal{T}$ we have that $g(A) \subseteq^* C$ and $C \cap (A \backslash g(A)) =^* \emptyset$. In this way, $C \cap A =^* g(A)$. We conclude that C trivializes g. $\qquad\square$

Proof of $\Leftarrow$. Again, by contra-positive. Suppose that there is C which trivializes g. Then $C \cap A =^* g(A)$ for any $A \in \mathcal{T}$. In particular $g(A) \subseteq^* C$ and $C \cap (A \backslash g(A)) = \emptyset$. We conclude that C separates $\mathcal{C}(\mathcal{T}, g)$. $\qquad\square$

$\blacksquare$

Definition 3.1.59. Let $\mathcal{T}$ be a $*$-lower semi-lattice over a set X. For $f, g \in C(\mathcal{T})$ we can define $f \cdot g \in C(\mathcal{T})$ given as:

$$(f \cdot g)(A) = f(A) \Delta g(A).$$

The following lemma is easy.

Lemma 3.1.60. $C(\mathcal{T})$ *is an abelian (Boolean)*[2] *group with respect to this operation and* $Tr(\mathcal{T})$ *is a subgroup of it.*

Since $C(\mathcal{T})$ is abelian, then $Tr(\mathcal{T})$ is a normal subgroup. Therefore, we can consider the quotient induced by it.

Definition 3.1.61 (Gap cohomology group)**.** We define the *gap cohomology group of* $\mathcal{T}$ as the quotient of these two groups. That is,

$$G(\mathcal{T}) = C(\mathcal{T})/Tr(\mathcal{T}).$$

We say that $f, g \in C(\mathcal{T})$ are *cohomologous* if the class of f is equal to the class of g inside $G(\mathcal{T})$, or equivalently, if $f \cdot g \in Tr(\mathcal{T})$.

Remark 3.1.62. If $\mathcal{T}$ has size ω_1, then $|G(\mathcal{T})| \leqslant 2^{\omega_1}$.

In [105], Daniel E. Talayco defined the gap cohomology group for the particular case of ω_1-towers. In there he showed that the $2^\omega \leqslant |G(\mathcal{T})|$ for any ω_1-tower $\mathcal{T}$. In order to do that, he proved the following Theorem.

Theorem 3.1.63 (The $\aleph_0$ gap theorem)**.** *let* $\mathcal{T} = \langle T_\alpha \rangle_{\alpha \in \omega_1}$ *be an* ω_1-*tower. There is a sequence of functions* $\langle f_\alpha \rangle_{\alpha \in \omega_1}$ *with the following properties:*

- $\forall \alpha \in \omega_1 \, (f_\alpha : T_\alpha \longrightarrow \omega)$,

- $\forall \alpha < \beta \in \omega_1 \, (f_\alpha|_{T_\alpha \cap T_\beta} =^* f_\beta|_{T_\alpha \cap T_\beta})$,

- $\forall m < n \in \omega \, ((f_\alpha^{-1}[\{m\}], f_\alpha^{-1}[\{n\}])_{\alpha \in \omega_1} \text{ is a Hausdorff gap })$.

---

[2]A group $(G, \cdot, e)$ is Boolean if $g \cdot g = e$ for any $g \in G$. Equivalently, G is a vector space over $\mathbb{Z}_2$. In particular, the isomorphism type of G is completely determined by its cardinality.

He also proved that the $\lozenge$-principle implies that the size of this group is always 2^{ω_1}. Later, Stevo Todorčević noted that the $\aleph_0$ gap theorem already gives you the previous conclusion without assuming any extra axioms (see Theorem 32 in [106]). Namely, if $2^\omega = 2^{\omega_1}$ the result is clear due to Talayco's calculations. On the other hand, if $2^\omega < 2^{\omega_1}$ then the result follows from an easy counting argument involving the quotient structure of $G(\mathcal{T})$. In [80], Charles Morgan constructed an ω_1-tower whose gap cohomology group can be explicitly calculated. This construction was carried through the use of morasses. Finally, in [36], Ilijas Farah improved the $\aleph_0$ gap theorem. This was achieved by considering the following concept.

Definition 3.1.64 (Coherent families of functions)**.** A *coherent family of functions* supported by an ω_1-tower $\langle T_\alpha \rangle_{\alpha \in \omega_1 \backslash \omega}$ is a family of functions $\langle f_\alpha \rangle_{\alpha \in \omega_1}$ such that:

(1) $\forall \alpha \in \omega_1 \backslash \omega \, (f_\alpha : T_\alpha \longrightarrow \alpha)$,

(2) $\forall \alpha, \beta \in \omega_1 \backslash \omega \, (f_\alpha|_{T_\alpha \cap T_\beta} =^* f_\beta|_{T_\alpha \cap T_\beta})$.

Given such family, we define L_α^ξ as $f_\alpha^{-1}[\{\xi\}]$ for all $\xi \in \omega_1$ and $\alpha > \xi$. Additionally, we let $\mathcal{L}^\xi = \langle L_\alpha^\xi \rangle_{\alpha > \xi}$

Lemma 3.1.65. *If $\langle f_\alpha \rangle_{\omega_1 \backslash \omega}$ is as in the previous definition and $\xi < \mu \in \omega_1$, the pair $(L_\alpha^\xi, L_\alpha^\mu)_{\alpha \in \omega_1 \backslash \mu}$ is a pregap. We will denote it as $(\mathcal{L}^\xi, \mathcal{L}^\mu)$ although, in principle, $\mathcal{L}^\xi = \langle L_\alpha^\xi \rangle_{\alpha > \xi}$ instead of $\langle L_\alpha^\xi \rangle_{\alpha > \mu}$.*

Definition 3.1.66. Let $\mathfrak{F} = \langle f_\alpha \rangle_{\alpha \in \omega_1 \backslash \omega}$ be a coherent family of functions supported by an ω_1-tower. We say that $\mathfrak{F}$ is Luzin (respectively Hausdorff) if $(\mathcal{L}^\xi, \mathcal{L}^\mu)$ is Luzin (respectively Hausdorff) for each $\xi < \mu \in \omega_1$.

Ilijas Farah constructed a Haudorff coherent family of functions supported by an ω_1-tower by forcing it and then appealing to Keisler's completeness Theorem for $L^\omega(Q)$. Unlike Talayco's result, the proof idea behind the construction of Farah does not work for any ω_1-tower. This is due to the nature of Keisler's Theorem.

Here we give a direct construction of a Luzin coherent family of functions with the use of a 2-construction scheme. No previous direct construction was known.

Theorem 3.1.67. *There is a Luzin coherent family of functions supported by an ω_1-tower.*

Proof. Let $\mathcal{F}$ be a 2-construction scheme. First we define ω_1-tower over the countable set

$$ N = \bigcup_{k \in \omega} N_k $$

where each N_k is equal to $\{k\} \times k \times r_k \times r_k$. Given $\alpha \in \omega_1 \backslash \omega$ we define T_α as

$$ \bigcup_{\Xi_\alpha(k) \geqslant 0} N_k = \{(k, s, i, j) \in \omega^4 \, : \, \Xi_\alpha(k) \geqslant 0 \text{ and } (s, i, j) \in k \times r_k \times r_k \}. $$

Note that for each $\alpha < \beta \in \omega_1$ and every $k > \rho(\alpha, \beta)$ we have that $T_\alpha \cap N_k \subseteq T_\beta \cap N_k$ due to the part (c) of Lemma 2.3.9. Consequently, $T_\alpha \subseteq^* T_\beta$. In this way $\mathcal{T} = \langle T_\alpha \rangle_{\alpha \in \omega_1 \backslash \omega}$ is a tower.

Now we define a Luzin coherent family of functions supported by $\mathcal{T}$. Let $\alpha \geqslant \omega$. Note that if $x \in T_\alpha$ then $x = (k, s, i, j)$ where $k > 0$, $\|\alpha\|_k > r_k$ and $i, j < r_k$ (In particular, $(\alpha)_k(i)$ and $(\alpha)_k(j)$ are defined). In this way, we can define $f_\alpha : T_\alpha \longrightarrow \alpha$ as follows:

$$f_\alpha(k, s, i, j) = \begin{cases} (\alpha)_k(i) & \text{if } \Xi_\alpha(k) = 0 \\ (\alpha)_k(j) & \text{if } \Xi_\alpha(k) = 1 \end{cases}$$

By definition, the family $\langle f_\alpha \rangle_{\alpha \in \omega_1 \setminus \omega}$ satisfies the point (1) of Definition 3.1.64. The two following claims will finish the proof.

<u>Claim 1:</u> $\langle f_\alpha \rangle_{\alpha \in \omega_1 \setminus \omega}$ satisfies the point (2) of Definition 3.1.64.

Proof of claim. Let $\alpha < \beta \in \omega_1 \setminus \omega$ and $(k, s, i, j) \in T_\alpha \cap T_\beta$ be such that $k > \rho(\alpha, \beta)$. By definition of T_α and T_β it follows that both $\Xi_\alpha(k)$ and $\Xi_\beta(k)$ are non-negative numbers. In virtue of the part (c) of Lemma 2.3.9, it must happen that $\Xi_\alpha(k) = \Xi_\beta(k)$. Furthermore, $(\alpha)_k \sqsubseteq (\beta)_k$ so $(\alpha)_k(i) = (\beta)_k(i)$ and $(\alpha)_k(j) = (\beta)_k(j)$. Hence, $f_\alpha(k, s, i, j, s) = f_\beta(k, s, i, j)$. Since all but finitely many elements of $T_\alpha \cap T_\beta$ have their first coordinate bigger than $\rho(\alpha, \beta)$, we have shown that $f_\alpha|_{T_\alpha \cap T_\beta} =^* f_\beta|_{T_\alpha \cap T_\beta}$. □

<u>Claim 2:</u> $\langle f_\alpha \rangle_{\alpha \in \omega_1 \setminus \omega}$ is Luzin.

Proof of claim. Let $\xi < \mu \in \omega_1$, $\beta > \mu$ and $n \in \omega$. We need to prove that the set

$$\{\alpha \in \beta \setminus \mu \, : \, |L_\alpha^\xi \cap L_\beta^\mu| \leqslant n\}$$

is finite. We claim that this set is contained in $(\beta)_l$ where $l = \max(n, \rho^{\{\xi, \mu, \beta\}})$. Indeed, take an $\alpha \in \beta \setminus \mu$ such that $\alpha \notin (\beta)_l$ and let $k = \rho(\alpha, \beta)$ (so obviously $l < k$). According to the part (b) of Lemma 2.3.9, $\Xi_\alpha(k) = 0$ and $\Xi_\beta(k) = 1$. Furthermore, since $\alpha \notin (\beta)_l$ then $k \geqslant \rho^{\{\xi, \mu, \beta\}}$. This means that both ξ and μ belong to $(\alpha)_k$. Hence

$$\Xi_\xi(k) \leqslant \Xi_\mu(k) \leqslant \Xi_\alpha(k) = 0 < \Xi_\beta(k).$$

By virtue of the part (c) of Lemma 2.3.9, $\Xi_\mu(k) = \Xi_\xi(k) = -1$. In other words, $\|\xi\|_k$ and $\|\mu\|_k$ are numbers strictly smaller than r_k. Moreover, $(\alpha)_k(\|\xi\|_k) = (\beta)_k(\|\xi\|_k) = \xi$ and $(\alpha)_k(\|\mu\|_k) = (\beta)_k(\|\mu\|_k) = \xi$. From this it follows that

$$\{k\} \times k \times \{\|\xi\|_k\} \times \{\|\mu\|_k\} \subseteq f_\alpha^{-1}[\{\xi\}] \cap f_\beta^{-1}[\{\mu\}] = L_\alpha^\xi \cap L_\beta^\mu.$$

As this set has cardinality k (which is bigger than n) we are done. □

∎

The gap cohomology group of ω_1-towers with a Luzin coherent family of functions can be explicitly calculated using Talayco's ideas. We present here such calculations for the sake of completeness.

Proposition 3.1.68. *Let $\mathcal{T} = \langle T_\alpha \rangle_{\alpha \in \omega_1}$ be an ω_1-tower and suppose there is a Luzin coherent family of functions $\langle f_\alpha \rangle_{\alpha \in \omega_1}$ supported by $\mathcal{T}$. Then $|G(\mathcal{T})| = 2^{\omega_1}$.*

Proof. For each $S \in \mathscr{P}(\omega_1)$, we define a coherent subsystem $g_S : \mathcal{T} \longrightarrow \mathscr{P}(\omega)$ as follows:

$$g_S(T_\alpha) = f_\alpha^{-1}[S].$$

Note that if $S, S' \in \mathscr{P}(\omega_1)$ then $g_S \cdot g_{S'} = g_{S \Delta S'}$. Let $\mathcal{S} \subseteq \mathscr{P}(\omega_1)$ be a family of size 2^{ω_1} so that $S \Delta S'$ and $\omega_1 \backslash (S \Delta S')$ are infinite for any two distinct $S, S' \in \mathcal{S}$. Given such S and S' we claim that g_S and $g_{S'}$ are not cohomologous. Indeed, let $\xi \in S \Delta S'$ and $\mu \in \omega_1 \backslash (S \Delta S')$ with $\xi < \mu$. Then $L_\alpha^\xi \subseteq g_{S \Delta S'}(T_\alpha)$ and $L_\alpha^\mu \subseteq g_{\omega_1 \backslash (S \Delta S')}(T_\alpha) = T_\alpha \backslash g_{S \Delta S'}(T_\alpha)$ for any $\alpha > \mu$. By hypothesis we have that $(L_\alpha^\xi, L_\alpha^\mu)_{\alpha \in \omega_1}$ is a Luzin-gap. This implies that $G(\mathcal{T}, g_{S \Delta S'})$ is a gap as well. That is, $g_{S \Delta S'} = g_S \cdot g_{S'}$ is a non-trivial coherent subsystem of $\mathcal{T}$. We have proved that for any two distinct $S, S' \in \mathcal{S}$, g_S and $g_{S'}$ are not cohomologous. Since $|S| = 2^{\omega_1}$, then $|G(\mathcal{T})| = 2^{\omega_1}$. ∎

As a second application of the concept of Luzin representations, we use Theorem 3.1.20 to generalize the results of Talayco, Todorčević, Morgan and Farah regarding the existence of "big gap cohomology groups". It is worth noting that, as for now, it is unclear if there is an analogous to the $\aleph_0$ gap theorem for aribrary $*$-lower semi-lattices. Therefore, we can not argue in the same way as Todorčević in order to conclude that the gap cohomology group has always cardinality 2^{ω_1}.

Theorem 3.1.69. *Let $(X, <, \wedge)$ be an ω_1-like lower semi-lattice. Then there is a $*$-lower semi-lattice $\mathcal{T}$ isomorphic to X with $|G(\mathcal{T})| = 2^{\omega_1}$.*

Proof. Consider $X \times 2$ with the order given by $(x, i) < (y, j)$ if and only if $x < y$ and $i = j$. It is easy to see that $X \times 2$ is also ω_1-like. In virtue of Theorem 3.1.20 we can take $\mathcal{T}' = \langle T'_{(x,i)} \rangle_{(x,i) \in X \times 2}$ and $\mathcal{A} = \langle A_{(x,i)} \rangle_{(x,i) \in X \times 2}$ so that $(\mathcal{T}', \mathcal{A})$ is a Luzin representation of $X \times 2$. For any $x \in X$ let $T_x = T'_{(x,0)} \cup T'_{(x,1)}$. It is straightforward that the following properties hold for any $x, y \in X$:

(a) $A_{(x,0)} \cup A_{(x,1)} \subseteq T_x$.

(b) If $y \nleqslant x$ then $A_{(y,0)} \cup A_{(y,1)} \subseteq^* T_y \backslash T_x$. In other words, $(A_{(y,0)} \cup A_{(y,1)}) \cap T_x =^* \emptyset$.

(c) $T_x \cap T_y =^* T_{x \wedge y}$.

In particular, properties (b) and (c) imply that $\mathcal{T} = \langle T_x \rangle_{x \in X}$ is a $*$-lower semi-lattice isomorphic to X. We claim that $|G(\mathcal{T})| = 2^{\omega_1}$. Trivially $|G(\mathcal{T})| \leqslant 2^{\omega_1}$, so we will only prove the other inequality. For this, let $S \in \mathscr{P}(X)$. . We will build recursively (using that X is well-founded) a coherent subsystem $g_S : \mathcal{T} \longrightarrow \mathscr{P}(\omega)$ such that $A_{(x,0)} \subseteq g_S(T_x)$ for any $x \in X$ and:

(A)

If $x \in S$, then $A_{(x,1)} \cap g_S(T_x) =^* \emptyset$.

(B)

If $x \notin S$, then $A_{(x,1)} \subseteq g_S(T_x)$.

Suppose that $y \in X$ and we have defined $g_S(T_x)$ for any $x < y$. By means of Lemma 3.1.58 we know that $\mathcal{G}(\mathcal{T}|_y, g_S|_{\mathcal{T}|_y})$ is pregap where $\mathcal{T}|_y = \langle T_x \rangle_{x < y}$. As there are no countable pregaps and $|(-\infty, y)| \leqslant \omega$, we conclude that there is $C \in \mathscr{P}(\omega)$ separating it. Since $g_S(T_x) \subseteq T_x \subseteq^* T_y$ for any $x < y$, we may assume without loss of generality

that $C \subseteq T_y$. Note that $C \cap T_x =^* g_S(T_x)$ for any $x < y$. We now define $g_S(T_y)$ by cases. If $y \in S$ define $g_S(T_y)$ as $(C \cup A_{(y,0)}) \setminus A_{(y,1)}$. Otherwise define $g_S(T_y)$ as $C \cup A_{(y,0)} \cup A_{(y,1)}$. Since both $A_{(y,0)}$ and $A_{(y,1)}$ are almost disjoint with T_x for all $x < y$ due to the point (b), it follows that $g_S(T_y) \cap T_x =^* C \cap T_x =^* g_S(T_x)$ for any such x. This finishes the recursion.

Now consider $\mathcal{S} \subseteq [X]^{\omega_1}$ a family of cardinality 2^{ω_1} so that $S \Delta S'$ is uncountable for any two distinct $S, S' \in \mathcal{S}$. The key fact needed to finish the proof is that if $x \in S \Delta S'$ then $A_{(x,1)} \subseteq g_S \cdot g_{S'}(T_x)$ and $A_{(x,0)} \subseteq T_x \setminus (g_S \cdot g_{S'}(T_x))$. In this way, any separation of the pregap $\mathcal{G}(\mathcal{T}, g_S \cdot g_{S'})$ would also separate the pregap $(A_{(x,1)}, A_{(x,0)})_{x \in S \Delta S'}$ which is impossible since $\mathcal{A}$ is a Luzin family and $S \Delta S'$ is uncountable. This shows that g_S and $g_{S'}$ are not cohomologous. Since $|\mathcal{S}| = 2^{\omega_1}$ then $|G(\mathcal{T})| = 2^{\omega_1}$. ∎

3.1.3   Destructibility of gaps

In this subsection we will study (ω_1, ω_1)-gaps in the context of forcing. The reader can find fairly complete treatments of this subject in [13], [91] and [113].

The following fundamental lemma is attributed to Kenneth Kunen. We will prove it for the sake of completeness. For others proofs see [37], [91] and [113].

Lemma 3.1.70. *Let $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-pregap with $L_\alpha \cap R_\alpha = \emptyset$ for any α. Then $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is a gap if and only if for any $S \in [\omega_1]^{\omega_1}$ there are $\alpha < \beta \in S$ such that $(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) \neq \emptyset$.*

Proof. Proof of $\Rightarrow$. Assume that $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is a gap and let $S \in [\omega_1]^{\omega_1}$. Suppose towards a contradiction that $(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) = \emptyset$ for all $\alpha, \beta \in S$. Let $C = \bigcup_{\beta \in S} L_\beta$. Given $\alpha \in \omega_1$ there is $\beta \in S$ such that $\alpha < \beta$. Note that $L_\alpha \subseteq^* L_\beta \subseteq C$ and $R_\alpha \cap C \subseteq^* R_\beta \cap C = \emptyset$. In this way C separates $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ which is a contradiction. Thus, the proof of this implication is over. □

Proof of $\Leftarrow$. Again, by contradiction. Suppose that $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is not a gap and let C be a set which separates it. By the pigeonhole principle we can find $S \in [\omega_1]^{\omega_1}$ and $a, b \in [\omega]^{<\omega}$ such that $L_\alpha \setminus C = a$ and $R_\alpha \cap C = b$ for any $\alpha \in S$. Note that if $\alpha \in S$ then $L_\alpha \cap b = \emptyset$ because $b \subseteq R_\alpha$ and $L_\alpha \cap R_\alpha = \emptyset$. Therefore, $L_\alpha \cap R_\beta \subseteq ((C \cup a) \setminus b) \cap R_\beta = \emptyset$ whenever $\alpha, \beta \in S$. With this contradiction we finish the proof. □

∎

Remark 3.1.71. For the rest of this section we will assume that if $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is an (ω_1, ω_1)-pregap then it is normal. That is, $L_\alpha \cap R_\alpha = \emptyset$ any α.

Definition 3.1.72 (Destructible gaps)**.** Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap. We say that $(\mathcal{L}, \mathcal{R})$ is *destructible* if there is a forcing notion $\mathbb{P}$ which preserves ω_1 in such way that $(\mathcal{L}, \mathcal{R})$ is not a gap in some generic extension through $\mathbb{P}$. If this does not happen, the gap is said to be *indestructible*.

Remark 3.1.73. The Hausdorff condition is absolute. Hence any Hausdorff gap is indestructible. Note that under PID any (ω_1, ω_1)-gap is undestructible. This is due to the Remark 3.1.8.

In Definition 3.1.32 we defined the forcing $\mathbb{P}(\mathcal{D}, \mathcal{E})$ where $(\mathcal{D}, \mathcal{E})$ is a pregap (not necesarily of type (ω_1, ω_1)) with $|\mathcal{D}| = |\mathcal{E}| = \omega_1$. According to Proposition 3.1.34, this forcing is *ccc* if and only if the pregap $(\mathcal{D}, \mathcal{E})$ can be separated in some ω_1-preserving extension of the universe. Forcings with such properties have already been studied for the case of (ω_1, ω_1)-gaps. In the following definition we present some well-known reincarnations of them (see [25], [91], [121] or [133]).

Definition 3.1.74. Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap indexed as $(L_\alpha, R_\alpha)_{\alpha \in X}$ where X is an uncountable set of ordinals. We define the following forcing notions:

- $\chi_0(\mathcal{L}, \mathcal{R}) = \{p \in [X]^{<\omega} : \left(\bigcup_{\alpha \in p} L_\alpha \right) \cap \left(\bigcup_{\alpha \in p} R_\alpha \right) = \emptyset \}$.

- $\chi_1(\mathcal{L}, \mathcal{R}) = \{p \in [X]^{<\omega} : \forall \alpha \neq \beta \in p \left((L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) \neq \emptyset \right) \}$.

both ordered by reverse inclusion.

Lemma 3.1.75. *Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap indexed as $(L_\alpha, R_\alpha)_{\alpha \in X}$ where X is an uncountable set of ordinals. Then $(\mathcal{L}, \mathcal{R})$ is a gap if and only if $\mathbb{P} = \chi_1(\mathcal{L}, \mathcal{R})$ is ccc.*

Proof. First suppose that $(\mathcal{L}, \mathcal{R})$ is a gap and consider $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$. By refining $\mathcal{A}$ we can assume that the following conditions hold:

- There is $n \in \omega$ such that $|p| = n$ for any $p \in \mathcal{A}$.

- There are $\sigma, \tau : n \times n \longrightarrow [\omega]^{<\omega}$ so that for any $i \leqslant j \in n$, $L_{p(i)} \backslash L_{p(j)} = \sigma(i, j) = \sigma(j, i)$ and $R_{p(i)} \backslash R_{p(j)} = \tau(i, j) = \tau(j, i)$.

- There is $\mu : n \times n \longrightarrow [\omega]^{<\omega}$ so that for any $(i, j) \in n \times n$, $L_{p(i)} \cap R_{q(j)} = \mu(i, j)$.

- $\mathcal{A}$ forms a root-tail-tail Δ-system with a root R of cardinality $r < n$.

By means of Lemma 3.1.70 we know that there are distinct $p, q \in \mathcal{A}$ so that

$$(L_{p(r)} \cap R_{q(r)}) \cup (L_{q(r)} \cap R_{p(r)}) \neq \emptyset.$$

Claim: $p \cup q \in \mathbb{P}$.

Proof of claim. It is enough to take $i, j < n$ and show that $(L_{p(i)} \cap R_{q(j)}) \cup (L_{q(j)} \cap R_{p(i)}) \neq \emptyset$. If $i < r$ then $p(i) = q(i)$ so the previous set is equal to $(L_{q(i)} \cap R_{q(j)}) \cup (L_{q(j)} \cap R_{q(i)})$ which by hypothesis is non-empty because $q \in \mathbb{P}$. An analogous argument applies when $j < r$ so let us assume that $r \leqslant i, j$. But if $i \neq j$ then

$$\emptyset \neq (L_{p(i)} \cap R_{p(j)}) \cup (L_{p(j)} \cap R_{p(i)}) = \mu(i, j) \cup \mu(j, i) \ \subseteq (L_{p(i)} \cap R_{q(j)}) \cup (L_{q(j)} \cap R_{p(i)}).$$

So we might as well assume that $i = j$. In this case we have that

$$(L_{p(r)} \cap R_{q(r)}) \cap \tau(r, i) \subseteq (L_{p(r)} \cap R_{q(r)}) \cap R_{p(r)} \subseteq R_{p(r)} \cap L_{p(r)} = \emptyset,$$

$$(L_{p(r)} \cap R_{q(r)}) \cap \sigma(r, i) \subseteq R_{q(r)} \cap L_{q(r)} = \emptyset.$$

Therefore $L_{p(r)} \cap R_{q(r)} \subseteq R_{q(r)} \backslash \tau(r, i) \subseteq R_{q(i)}$ and $L_{p(r)} \cap R_{q(r)} \subseteq L_{p(r)} \backslash \sigma(r, i) \subseteq L_{p(i)}$. That is, $L_{p(r)} \cap R_{q(r)} \subseteq L_{p(i)} \cap R_{q(j)}$. In the same way we can show that $L_{p(r)} \cap R_{q(r)} \subseteq L_{q(i)} \cap R_{p(i)}$ so

$$\emptyset \neq (L_{p(r)} \cap R_{q(r)}) \cup (L_{q(r)} \cap R_{p(r)}) \subseteq (L_{p(i)} \cap R_{q(i)}) \cup (L_{q(i)} \cap R_{p(i)}).$$

$\square$

This finishes the proof of this implication. The other one is direct from Lemma 3.1.70. ∎

Lemma 3.1.76. *Let $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-pregap indexed as $(L_\alpha, R_\alpha)_{\alpha \in X}$ where X is an uncountable set of ordinal. Then $\mathbb{P} = \chi_0(\mathcal{L}, \mathcal{R})$ is ccc if and only if for any $S \in [X]^{\omega_1}$ there are $\alpha < \beta$ such that $(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) = \emptyset$.*

Proof. Proof of $\Leftarrow$. Take $S \in [X]^{\omega_1}$. Then $\mathcal{A} = \{\{\alpha\} : \alpha \in S\}$ is an uncountable subset of $\mathbb{P}$. By hypothesis there are distinct $\alpha, \beta \in S$ so that $\{\alpha\}$ and $\{\beta\}$ are compatible. It is easy to see that $(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) = \emptyset$. $\square$

Proof of $\Rightarrow$. Let $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$. We can suppose without loss of generality that $\mathcal{A}$ satisfies the same conditions as in Lemma 3.1.75 for some σ, τ, μ and R. Note that the set $S = \{p(n-1) : p \in \mathcal{A}\}$ is uncountable so by the hypotheses we can find distinct p and q so that

$$(L_{p(n-1)} \cap R_{q(n-1)}) \cup (L_{q(n-1)} \cap R_{p(n-1)}) = \emptyset.$$

Again, we claim that $p \cup q \in \mathbb{P}$. For this it suffices to prove that if $i, j < n$ then $L_{p(i)} \cap R_{q(j)} = \emptyset$. Indeed, for any such i, j it happens that

$$L_{p(i)} \subseteq L_{p(n-1)} \cup \sigma(i, n-1) \subseteq L_{p(n-1)} \cup L_{q(i)}.$$

Therefore $L_{p(i)} \cap (R_{q(n-1)} \subseteq L_{p(n-1)}) \cap R_{q(n-1)} \cup (L_{q(i)} \cap R_{q(n-1)}) = \emptyset$. In an analogous way $R_{q(j)} \subseteq R_{q(n-1)} \cup R_{p(j)}$. Hence, we conclude that

$$L_{p(i)} \cap R_{q(j)} \subseteq (L_{p(i)} \cap R_{q(n-1)}) \cup (L_{p(i)} \cap L_{p(j)}) = \emptyset.$$

$\square$

∎

The following theorem can be found in [25], [91], [121] and [133].

Theorem 3.1.77. *Let $(\mathcal{A}, \mathcal{B})$ be an (ω_1, ω_1)-pregap:*

- $\chi_1(\mathcal{L}, \mathcal{R})$ *is ccc if and only if $(\mathcal{L}, \mathcal{R})$ is a gap. In this case, there is some condition in $\chi_1(\mathcal{L}, \mathcal{R})$ forcing $(\mathcal{L}, \mathcal{R})$ to be indestructible.*

- $\chi_0(\mathcal{L}, \mathcal{R})$ *is ccc if and only if $(\mathcal{L}, \mathcal{R})$ is destructible. In this case, there is some condition in $\chi_0(\mathcal{L}, \mathcal{R})$ forcing $(\mathcal{L}, \mathcal{R})$ to be separated.*

A particular instance of destructible gaps are the ones satifying the following condition.

Definition 3.1.78 (Todorčević condition)**.** Let $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-pregap. We say that $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ is *Todorčević* if for any $S \in [\omega_1]^{\omega_1}$ there are $\alpha < \beta$ such that $L_\alpha \subseteq L_\beta$ and $R_\alpha \subseteq R_\beta$.

Fulgencio Lopez and Stevo Todorčević showed that the existence of Todorčević gaps follows from CA_3 and $CA_2(part)$. Here we give another proof of such result based on the construction given in Theorem 3.1.10.

Theorem 3.1.79 (Under $CA_2(part)$)**.** *Let $\mathcal{F}$ be a $\mathcal{P}$-2-capturing 2-construction scheme with $\mathcal{P} = \{P_0, P_1\}$. Also, consider $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ the Hausdorff gap constructed in Theorem 3.1.10. Let*

$$C = \bigcup \{ \{2k, 2k+1\} : k \in P_0 \backslash 1 \}$$

and define

$$L_\alpha^C = L_\alpha \cap C = \{2k + \Xi_\alpha(k) : k \in P_0 \backslash 1 \text{ and } \Xi_\alpha(k) \geqslant 0\},$$

$$R_\alpha^C = R_\alpha \cap C = \{2k + (1 - \Xi_\alpha(k)) : k \in P_0 \backslash 1, \text{ and } \Xi_\alpha(k) \geqslant 0\}$$

for each $\alpha \in \omega_1$. Then $(\mathcal{L}|_C, \mathcal{R}|_C) = (L_\alpha^C, R_\alpha^C)_{\alpha \in \omega_1}$ is a Todorčević gap.

Proof. $(L_\alpha^C, R_\alpha^C)_{\alpha \in \omega_1}$ is a pregap by similar reasons as the ones in the proof of Theorem 3.1.10. That is, $L_\alpha^C \cap R_\alpha^C = \emptyset$ for each $\alpha \in \omega_1$ and if $\alpha < \beta$ then:

(a) $L_\alpha^C \backslash L_\beta^C \subseteq \{2k + \Xi_\alpha(k) : k \leqslant \rho(\alpha, \beta) \text{ and } \Xi_\alpha(k) \neq \Xi_\beta(k)\}$,

(b) $R_\alpha^C \backslash R_\beta^C \subseteq \{2k + (1 - \Xi_\alpha(k)) : k \leqslant \rho(\alpha, \beta) \text{ and } \Xi_\alpha(k) \neq \Xi_\beta(k)\}$.

In particular, as a consequence of (a) we have:

(c) $L_\alpha^C \cap R_\beta^C \subseteq \{2k + \Xi_\alpha(k) : k \leqslant \rho(\alpha, \beta) \text{ and } \Xi_\alpha(k) \neq \Xi_\beta(k)\}$.

In order to prove that $(L_\alpha^C, R_\alpha^C)_{\alpha \in \omega_1}$ is a gap we will use the equivalence provided by Lemma 3.1.70. Let $S \in [\omega_1]^{\omega_1}$. Since $\mathcal{F}$ is $\mathcal{P}$-2-capturing there are $\alpha < \beta \in S$ so that $\{\alpha, \beta\}$ is captured at some level $l \in P_0$. By Proposition 2.4.10 it follows that $\Delta(\alpha, \beta) = l = \rho(\alpha, \beta)$. Hence, $\Xi_\alpha(k) = \Xi_\beta(k)$ for any $k < l$. From this and by the the point (c) above we deduce that $L_\alpha^C \cap R_\beta^C \subseteq \{2l + \Xi_\alpha(l)\}$. Furthermore, since $l = \rho(\alpha, \beta)$ and $\mathcal{F}$ is a construction scheme(or just by point (2) in Proposition 2.4.10) we also have that $\Xi_\alpha(l) = 0$ and $\Xi_\beta(l) = 1$. Therefore we conclude that $L_\alpha^C \cap R_\beta^C = \{2k\}$, so in particular $(L_\alpha^C \cap R_\beta^C) \cup (L_\beta^C \cap R_\alpha^C) \neq \emptyset$. As we said before, by Lemma 3.1.70 we are done.

Now we will prove that $(L_\alpha^C, R_\alpha^C)_{\alpha \in \omega_1}$ is Todorčević. Let $S \in [\omega_1]^{\omega_1}$. Since $\mathcal{F}$ is $\mathcal{P}$-2-capturing there are $\alpha < \beta \in S$ such that $\{\alpha, \beta\}$ is captured at some level $l \in P_1$. Again, by Proposition 2.4.10 and by the points (a) and (b) written at the beginning of the proof we deduce that $L_\alpha^C \backslash L_\beta^C \subseteq \{2l\}$ and $R_\alpha^C \backslash R_\beta^C \subseteq \{2l + 1\}$. But $l \in P_1$ so $\{2l, 2l+1\}$ has empty intersection with both L_α^C and R_α^C by definition of these two sets. In this way we conclude that $L_\alpha^C \backslash L_\beta^C = \emptyset$ and $R_\alpha^C \backslash R_\beta^C = \emptyset$. In other words, $L_\alpha^C \subseteq L_\beta^C$ and $R_\alpha^C \subseteq R_\beta^C$. ∎

It is natural to ask whether destructible gaps can be constructed only by using a 2-capturing construction scheme. In the next theorem we will show that this is not the case. But first, let us note that the proofs of both Lemma 3.1.76 and Lemma 3.1.75 actually yield the two following results.

Lemma 3.1.80. *Let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap. Suppose that $\mathcal{A} \in [\chi_0(\mathcal{L}, \mathcal{R})]^{\omega_1}$. Then there are $n \in \omega$ and $\mathcal{B} \in [\mathcal{A}]^{\omega_1} \cap \mathscr{P}([\omega_1]^n)$ an uncountable root-tail-tail Δ-system such that for any two distinct $p, q \in \mathcal{B}$, the following conditions are equivalent:*

- *p is compatible with q.*

- *$(L_{p(n-1)} \cap R_{q(n-1)}) \cup (L_{q(n-1)} \cap R_{p(n-1)}) = \emptyset$. In other words, $\{p(n-1)\}$ is compatible with $\{q(n-1)\}$.*

Lemma 3.1.81. *Let $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$ be an (ω_1, ω_1)-gap. Suppose that $\mathcal{A} \in [\chi_1(\mathcal{L}, \mathcal{R})]^{\omega_1}$. Then there are $n, r \in \omega$ and $\mathcal{B} \in [\mathcal{A}]^{\omega_1} \cap \mathscr{P}([\omega_1]^n)$ a root-tail-tail Δ-system with root R of cardinality r such that for any two distinct $p, q \in \mathcal{B}$, the following conditions are equivalent:*

- *p is compatible with q.*

- *$(L_{p(r)} \cap R_{q(r)}) \cup (L_{q(r)} \cap R_{p(r)}) \neq \emptyset$. In other words, $\{p(r)\}$ is compatible with $\{q(r)\}$.*

Theorem 3.1.82. *Let $\mathcal{F}$ be a 2-capturing construction scheme. If $\mathfrak{m}_\mathcal{F} > \omega_1$, then there are no destructible gaps.*

Proof. Let us assume towards a contradiction that there is a destructible gap $(\mathcal{L}, \mathcal{R}) = (L_\alpha, R_\alpha)_{\alpha \in \omega_1}$. The forcing $\chi_0(\mathcal{L}, \mathcal{R})$ is never Knaster. Therefore, by Propositions 4.1.3 and 4.0.3, there are $X' \in [\chi_0(\mathcal{L}, \mathcal{R})]^{\omega_1}$ and an injective function $\zeta' : X \longrightarrow \omega_1$ in such way that for any two distinct $p, q \in X'$, if $\{\zeta'(p), \zeta'(q)\}$ is captured, then p and q are incompatible in $\chi_0(\mathcal{L}, \mathcal{R})$. By virtue of the Lemma 3.1.80, we may assume that the elements X' are singletons. In this way, we can define $X = \{\alpha \in \omega_1 : \{\alpha\} \in X'\}$ and $\zeta : X \longrightarrow \omega_1$ given by $\zeta(\alpha) = \zeta'(\{\alpha\})$. Note that for any two distinct $\alpha, \beta \in X$, if $\{\zeta(\alpha), \zeta(\beta)\}$ is captured, then

$$(L_\alpha \cap R_\beta) \cup (L_\alpha \cap R_\beta) \neq \emptyset.$$

<u>Claim</u>: $\chi_1(L_\alpha, R_\alpha)_{\alpha \in X}$ is *ccc* and *n*-preserves $\mathcal{F}$.

Proof of claim. We will prove the claim by appealing to the Lemma 4.0.3. Let $\mathcal{A}$ be an uncountable subset of $\chi_1(L_\alpha, R_\alpha)_{\alpha \in X}$ and $\nu : \mathcal{A} \longrightarrow \omega_1$ be an injective function. According to the Lemma 3.1.81, we may suppose that there are $n, r \in \omega$ so that $\mathcal{A} \in [X]^n$ and it forms a root-tail-tail Δ-system with root R of cardinality r so that for any two distinct $p, q \in \mathcal{A}$, if $(L_{p(r)} \cap R_{q(r)}) \cup (L_{q(r)} \cap R_{p(r)}) \neq \emptyset$, then p and q are compatible. Given $p \in \mathcal{A}$, let us define

$$A_p = \{\zeta(p(r)), \nu(p)\},$$

$$\alpha_p = \max(A_p).$$

Since both ζ and ν are injective functions and $p(r) \neq q(r)$ whenever $p \neq q$, we may assume that the A_p's are pairwise disjoint. Even more,we can suppose that there are $k, a \in \omega$ so that the following conditions hold for any two distinct $p, q \in \mathcal{A}$:

(a) $\rho^{A_p} = k$.

(b) $\|\alpha_p\|_k = a$.

(c) If $h : (\alpha_p)_k \longrightarrow (\alpha_q)_k$ is the increasing bijection $h(\nu(p)) = \nu(q)$ and $h(\zeta(p(r))) = \zeta(q(r))$.

(d) $\rho(\alpha_p, \alpha_q) > k$.

Since $\mathcal{F}$ is assumed to be 2-capturing, there are distinct $p_0, p_1 \in \mathcal{A}$ for which $\{\alpha_{p_0}, \alpha_{p_1}\}$ is captured. By virtue of the Lemma 2.4.11 and the points (a), (b), (c) and (d) above, are $\{\nu(p_0), \nu(p_1)\}$ and $\{\zeta(p_0(r)), \zeta(p_1(r))\}$ are also captured. In particular, this means that $(L_{p_0(r)} \cap R_{p_1(r)}) \cup (L_{p_1(r)} \cap R_{p_0(r)}) \neq \emptyset$. Thus, p_1 and p_1 are compatible. This finishes the proof.

$\square$

By the previous claim $\chi_1(L_\alpha, R_\alpha)_{\alpha \in X}$ is an uncountable *ccc* forcing which 2-preserves $\mathcal{F}$. Since $\mathfrak{m}_\mathcal{F} > \omega_1$, it follows that $(L_\alpha, R_\alpha)_{\alpha \in X}$ is an indestructible gap due to the Lemma 3.1.77 (hence, so is $(\mathcal{L}, \mathcal{R})$). This is a contradiction, so the proof is over. $\blacksquare$

Remark 3.1.83. Under the hypotheses of the Theorem 3.1.79 , define $L_\alpha^{\omega \setminus C} = L_\alpha \cap (\omega \setminus C)$ and $R_\alpha^{\omega \setminus C} = R_\alpha \cap (\omega \setminus C)$ for each $\alpha \in \omega_1$. A similar arguing yields that $(\mathcal{L}|_{\omega \setminus C}, \mathcal{R}|_{\omega \setminus C}) = (L_\alpha^{\omega \setminus C}, R_\alpha^{\omega \setminus C})_{\alpha \in \omega_1}$ is also a Todorčević gap.

Recall that the product of two forcings, say $\mathbb{P}$ and $\mathbb{Q}$, is *ccc* if and only if $\mathbb{P}$ is *ccc* and $\mathbb{P} \Vdash$ " $\mathbb{Q}$ *is ccc* ". Suppose we are in the particular situation where have $(\mathcal{L}_0, \mathcal{R}_0)$ and $(\mathcal{L}_1, \mathcal{R}_1)$ are two gaps such that $\chi_i(\mathcal{L}_0, \mathcal{R}_0) \times \chi_j(\mathcal{L}_1, \mathcal{R}_1)$ is *ccc* for any $i, j \in 2$. According to the previous theorem, destroying or making indestructible one of these two gaps by any of the previous forcings will not destroy or make indestructible the remaining one. This motivates the following definition.

Definition 3.1.84 (Independent families of gaps)**.** We say that a family $\langle (\mathcal{L}^c, \mathcal{R}^c) \rangle_{c \in I}$ of (ω_1, ω_1)-pregaps is *independent* if:

$$\prod_{c \in I}^{FS} \chi_{\phi(c)}(\mathcal{L}^c, \mathcal{R}^c)$$

is *ccc* for any $\phi : I \longrightarrow 2$. Additionally, we say that a coherent family of functions $\mathfrak{F}$ supported by an ω_1-tower is *independent* if the family $\langle (\mathcal{L}^{c(0)}, \mathcal{L}^{c(1)}) \rangle_{c \in [\omega_1 \setminus \omega]^2}$ is independent.

Remark 3.1.85. If $\langle (\mathcal{L}^c, \mathcal{R}^c) \rangle_{c \in I}$ is an independent family of (ω_1, ω_1)-pregaps, then $(\mathcal{L}^c, \mathcal{R}^c)$ is a gap for any $c \in I$.

Theorem 3.1.86 (Under $CA_2(part)$)**.** *There are two (ω_1, ω_1)-gaps which are Todorče-vić but not independent.*

Proof. Let $\mathcal{F}$ be a 2-capturing 2-construction scheme and consider the gaps $(\mathcal{L}|_C, \mathcal{R}|_C)$ and $(\mathcal{L}|_{\omega \setminus C}, \mathcal{R}|_{\omega \setminus C})$ defined in Theorem 3.1.79 and Remark 3.1.83 respectively. These two gaps are Todorčević, so we only need to prove that they are not independent. This will be done in the following claim.

<u>Claim:</u> $\mathbb{P} = \chi_0(\mathcal{L}|_C, \mathcal{R}|_C) \times \chi_0(\mathcal{L}|_{\omega \setminus C}, \mathcal{R}|_{\omega \setminus C})$ is not *ccc*.

Proof of claim. Suppose towards a contradiction that the forcing $\mathbb{P}$ is *ccc*. By Theorem 3.1.77, it follows that there is G a $\mathbb{P}$-generic filter over V so that both of $(\mathcal{L}|_C, \mathcal{R}|_C)$ and $(\mathcal{L}|_{\omega \setminus C}, \mathcal{R}|_{\omega \setminus C})$ are separated in $V[G]$ by some C_0 and C_1 respetively. In order to finish, just note that $C_0 \cup C_1$ separates the Hausdorff gap $(L_\alpha, R_\alpha)_{\alpha \in \omega_1}$. This is a contradiction since Hausdorff gaps undestructible. Thus, the proof is over.   □

■

In [3], Uri Abraham and Saharon Shelah used large independent families of Suslin trees in order to code certain sets of reals. Independent families of gaps can be used in a similar way. In [133], Teruyuki Yorioka proved, assuming the $\Diamond$-principle, that there is an independent family of 2^{ω_1} gaps. In here we will construct an independent coherent family of functions from FCA. For this purpose, we will need the following proposition (see [113]).

Proposition 3.1.87. *Let $\langle (\mathcal{L}^c, \mathcal{R}^c) \rangle_{c \in I}$ be a finite family of (ω_1, ω_1)-gaps indexed as $(L_\alpha^c, R_\alpha^c)_{\alpha \in X_c}$ for each $c \in I$. Also let $\phi : I \longrightarrow 2$. Suppose that for any uncountable $\mathcal{B} \subseteq \prod\limits_{i \in I} X_i$ there are distinct $g, h \in \mathcal{B}$ so that for any $c \in I$, if $g(c) \neq h(c)$ then:*

$$(L_{g(c)}^c \cap R_{h(c)}^c) \cup (L_{h(c)}^c \cap R_{g(c)}^c) = \emptyset \text{ if } \phi(c) = 0,$$

$$(L_{g(c)}^c \cap R_{h(c)}^c) \cup (L_{h(c)}^c \cap R_{g(c)}^c) \neq \emptyset \text{ if } \phi(c) = 1.$$

Then $\mathbb{P} = \prod\limits_{c \in I} \chi_{\phi(c)}(\mathcal{L}^c, \mathcal{R}^c)$ is ccc.

Proof. Let $\mathcal{A}$ be an uncountable subset of $\mathbb{P}$. For any $c \in I$ consider $\mathcal{A}^c = \{ p(c) : p \in \mathcal{A} \}$. Note that $\mathcal{A}^c \subseteq \chi_{\phi(c)}(\mathcal{L}^c, \mathcal{R}^c)$. In this way we can suppose that each $\mathcal{A}^c$ satisfies the conditions listed in the proof of Lemma 3.1.75 for some $n_c, \sigma_c, \tau_c, \mu_c, R_c$ and r_c. For any $p \in \mathcal{A}$ define $g_p \in \prod\limits_{c \in I} X_c$ as:

$$g_p(i) = \begin{cases} p(c)(n_c - 1) & \text{if } \phi(c) = 0 \\ p(c)(r_c) & \text{if } \phi(i) = 1 \end{cases}$$

By the hypotheses there are distinct $p, q \in \mathcal{A}$ such that for any $c \in I$, if $g_p(c) \neq g_q(c)$ then

$$(L_{g_p(c)}^c \cap R_{g_q(c)}^c) \cup (L_{g_q(c)}^c \cap R_{g_p(c)}^c) = \emptyset \text{ if } \phi(i) = 0,$$

$$(L^c_{g_p(c)} \cap R^c_{g_q(c)}) \cup (L^c_{g_q(c)} \cap R^c_{g_p(c)}) \neq \emptyset \ \textit{if } \phi(c) = 1.$$

By arguing on each coordinate in the same way as we did in the proof of Lemmas 3.1.76 an 3.1.75 we conclude that the function $p' \in \prod_{c \in I} [X_c]^{<\omega}$ given by $p'(c) = p(c) \cup q(c)$ for each $c \in I$, is an element of $\mathbb{P}$ which is obviously smaller than both p and q. This finishes the proof. ∎

Theorem 3.1.88 (FCA). *There is an independent coherent family of functions supported by an ω_1-tower.*

Proof. The construction here is similar to the one in Theorem 3.1.69. Fix a type $\langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$ such that $n_{k+1} > 2^{r^2_{k+1}}$ for all $k \in \omega$, and let $\mathcal{F}$ be a fully capturing construction scheme of that type. For each $k > 0$, let

$$N_k = \{k\} \times [r_k]^2,$$

$$N = \bigcup_{k>0} N_k.$$

Also enumerate $\mathscr{P}([r_k]^2)$ (possibly with repetitions) as $\langle S^k_i \rangle_{i<n_k}$ in such way that $S^k_0 = S^k_1 = [r_k]^2$. We start by defining an ω_1-tower over N. Given $\alpha \geqslant \omega$ and $k > 0$ we define $T^k_\alpha \subseteq N_k$ as follows:

$$T^k_\alpha = \begin{cases} \emptyset & \textit{if } \Xi_\alpha(k) = -1 \\ \{k\} \times S^k_i & \textit{if } \Xi_\alpha(k) = i \geqslant 0 \end{cases}$$

Finally, let $T_\alpha = \bigcup_{k \in \omega \setminus 1} T^k_\alpha$. In order to prove that $\mathcal{T} = \langle T_\alpha \rangle_{\alpha \in \omega_1 \setminus \omega}$ is in fact an ω_1-tower just note that if $\alpha < \beta \in \omega_1 \setminus \omega$ and $k > \rho(\alpha, \beta)$ then $T^k_\alpha \subseteq T^k_\beta$ by means of the point (c) of Lemma 2.3.9.

Now we will construct a coherent family of functions supported by $\mathcal{T}$. For this, let $\beta \geqslant \omega$. Note that if $x \in T_\beta$ then $x = (k, s)$ where $k > 0$, $\Xi_k(\beta) \geqslant 0$ and $s \in S^k_{\Xi_\beta(k)} \subseteq [r_k]^2$. In this way, we can define $f_\beta : T_\beta \longrightarrow \beta$ as:

$$f_\beta(k, s) = \begin{cases} (\beta)_k(s(0)) & \textit{if } \Xi_\beta(k) = 0 \\ (\beta)_k(s(1)) & \textit{if } \Xi_\beta(k) > 0 \end{cases}$$

It is easy to check that $\langle f_\alpha \rangle_{\alpha \in \omega_1 \setminus \omega}$ is a coherent family of functions supported by $\mathcal{T}$. This is done, again, by appealing to the point (c) of Lemma 2.3.9.

The only thing left to show is that $\langle f_\alpha \rangle_{\alpha \in \omega_1 \setminus \omega}$ is independent. Recall that a finite support product of forcings is *ccc* if and only if each finite subproduct is *ccc*. Let I be a non-empty finite subset of $[\omega_1 \setminus \omega]^2$ and let $\phi : I \longrightarrow 2$. We will finish by proving the following claim.

<u>Claim</u>: $\mathbb{P} = \prod_{c \in I} \chi_{\phi(c)}(\mathcal{L}^{c(0)}, \mathcal{L}^{c(1)})$ is *ccc*.

Proof of claim. The proof of this claim will be performed by appealing to the equivalence provided by Proposition 3.1.87. First note that for any $c \in I$, $(\mathcal{L}^{c(0)}, \mathcal{L}^{c(1)}) = (L^{c(0)}_\alpha, L^{c(1)}_\alpha)_{\alpha \in X_c}$ where $X_c = \omega_1 \setminus (c(1) + 1)$. Now, let $\mathcal{B}$ be an uncountable subset of $\prod_{c \in I} X_c$. For any $g \in \mathcal{B}$, define $D_g = \bigcup I \cup im(g)$. We need to prove the following claim.

(1) If $g, h \in B$ then $|im(g)| = |im(h)|$. Furthermore, if $\psi : im(g) \longrightarrow im(h)$ is the increasing bijection then $\psi(g(c)) = h(c)$ for any $c \in I$.

(2) $\{D_g : g \in \mathcal{B}\}$ is a root-tail-tail Δ-system with root $\bigcup I$.

Fix $k > \rho^{\cup I}$. Since $\mathcal{F}$ is a fully capturing construction scheme, there is $l > k$ and there are distinct $g_0, \ldots, g_{n_l - 1} \in \mathcal{B}$ so that the family $\{D_{g_0}, \ldots, D_{g_{n_l - 1}}\}$ is a captured at level l. According to (2) and by Lemma 2.4.5, $\Xi_{\cup I}(l) = -1$. In this way, the set $S = \{ \{\|c(0)\|_l, \|c(1)\|_l\} : c \in I \text{ and } \phi(c) = 1 \}$ is a subset of $[r_l]^2$. Therefore we can find $i < n_l$ so that $S_i^l = g_i$. The proof will be over once we show the following subclaim.

<u>Subclaim:</u> g_0 and g_i satisfy the hypotheses of Proposition 3.1.87. That is, for any $c \in I$, if $g_0(c) \neq g_i(c)$ then:[3]

$$(L_{g_0(c)}^{c(0)} \cap L_{g_i(c)}^{c(1)}) \cup (L_{g_i(c)}^{c(0)} \cap L_{g_0(c)}^{c(1)}) = \emptyset \text{ if } \phi(c) = 0,$$

$$(L_{g_0(c)}^{c(0)} \cap L_{g_i(c)}^{c(1)}) \cup (L_{g_i(c)}^{c(0)} \cap L_{g_0(c)}^{c(1)}) \neq \emptyset \text{ if } \phi(c) = 1.$$

Proof of subclaim. Let $c \in I$. Then $g(c) \neq h(c)$ due to condition (2). Furthermore, $\rho(g(c), h(c)) = l = \Delta(g(c), h(c))$ for any $c \in I$. This follows from condition (1) together with the fact that D_{g_0} is strongly isomorphic to D_{g_i}. By the definitions of $f_{g_0(c)}$ and $f_{g_i(c)}$ and the points (a) and (c) of Lemma 2.3.9, it follows that

$$L_{g_0(c)}^{c(0)} \cap L_{g_i(c)}^{c(1)} = \left(f_{g_0(c)}^{-1}[\{c(0)\}] \cap f_{g_i(c)}^{-1}[\{c(1)\}] \right) \cap T_{g_0(c)}^l.$$

On one hand, since $\Xi_{g_0(c)}(l) = 0$ then

$$f_{g_0(c)}^{-1}[\{c(0)\}] \cap T_{g_0(c)}^l = \{l\} \times \{s \in [r_l]^2 : s(0) = \|c(0)\|_l \}.$$

On the other hand, since $\Xi_{g_i(c)}(l) > 0$ and $S = S_i^l$ then

$$f_{g_0(c)}^{-1}[\{c(1)\}] \cap T_{g_0(c)}^l = \{l\} \times \{s \in S : s(1) = \|c(1)\|_l \}.$$

Thus, $L_{g_0(c)}^{c(0)} \cap L_{g_i(c)}^{c(1)} \subseteq \{ \{\|c(0)\|_l, \|c(1)\|_l\} \}$. Furthermore, such intersection is non-empty if and only if $\phi(c) = 1$ by the definition of S. By arguing in a similar way and using that $\|c(0)\|_l < \|c(1)\|_l$ we conclude that $L_{g_i(c)}^{c(0)} \cap L_{g_0(c)}^{c(1)}$ is always empty. This finishes the proof of the subclaim.

<div align="right">□</div>

<div align="right">□</div>

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[3]In this case $(\mathcal{L}^c, \mathcal{R}^c) = (L_\alpha^{c(0)}, L_\alpha^{c(1)})_{\alpha \in X_c}$. Therefore $L_{g_0(c)}^c = L_{g_0(c)}^{c(0)}$, $R_{g_i(c)}^c = L_{g_i(c)}^{c(1)}$, $L_{g_i(c)}^c = L_{g_i(c)}^{c(0)}$ and $R_{g_0(c)}^c = L_{g_0(c)}^{c(1)}$.

3.2 Trees and lines

The purpose of this section is to study uncountable trees and lines (totally ordered sets) from the point of view of construction schemes. Before dealing with any specific notion, it is convenient to recall the definition of a tree. Complete introductions to trees, lines and the duality between them can be found in [54], [64] and [114].

Definition 3.2.1 (Tree). A partial ordered set $(T, <)$ is called a *tree* if $(-\infty, t)_T$ is well-ordered for any $t \in T$.

When dealing with trees, we apply some changes in the notation related to partial orders. In this realm, intervals $(-\infty, t)_T$ and $(t, \infty)_T$ are denoted as $t_{\downarrow_T}$ and $t_{\uparrow_T}$ respectively. As usual, whenever there is no risk of confusion we will write those sets simply as $t_\downarrow$ and $t_\uparrow$ respectively.

It is easy to see each tree T is well-founded. Thus, according to the Section 1.2, there is a unique function $rank_T : T \longrightarrow Ord$ so that:

$$rank_T(t) = \sup(\, rank_T(s) + 1 \, ,: \ s < t \,).$$

Due to uniqueness, it can be shown that $rank_T(t) = ot(t_{\downarrow_T})$ for all $t \in T$. In this way, for each ordinal α, the level α of T can be described as $T_\alpha = \{\, t \in T \ : \ ot(t_\downarrow) = \alpha \,\}$. Given an ordinal β and $t \in T$, we define

$$T|_\beta = \bigcup_{\alpha < \beta} T_\alpha,$$

$$T|_t = \{\, s \in T \ : \ s \text{ is comparable with } t\}.$$

Finally, we say that $B \subseteq T$ is a *branch* if it is a maximal chain in T.

3.2.1 Countryman lines and Aronszajn trees

Countryman lines are certain kind of linear orders whose existence was proposed by Roger Simmons Countryman in the 1970's. In [93], Saharon Shelah constructed for the first time Countryman lines without appealing to any extra axioms. Years later, Stevo Todorčević presented in [120] an easy construction of such a line using walks on ordinals. The reader may find some variations of Todorčević original proof in [52] and [110].

Definition 3.2.2 (Countryman line). We say that a totally ordered set $(C, <)$ is a Countryman line if C is uncountable and C^2 can be covered by countably many chains[4].

In [75], Justin Moore put Countryman lines into the realm of canonical objects inside Mathematics. Specifically, he proved the following theorem.

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[4]Here, we consider the order over C^2 given by $(x, y) \leqslant (w, z)$ if and only $x \leqslant w$ and $y \leqslant z$.

Theorem 3.2.3 (Under PFA). *Let C be any Countryman line and X be a fixed subset of $\mathbb{R}$ of cardinality ω_1. Then any uncountable order contains an isomorphic copy of one of the following five orders:*

$$X,\ \omega_1,\ \omega_1^*,\ C,\ \text{or } C^*.$$

Where ω_1^* and C^* denote ω_1 and C but with the reverse order.

We will now define a Countryman line using construction schemes. It should be noted that in Chapter 3 of [110], Todorčević showed that ordinal metrics can also be used to define Countryman lines. Although working with construction schemes is equivalent to working with their induced ordinal metric, the definition of the line appearing in this thesis differs from the ones in [110].

The following proposition will help by giving us a better picture of the next definitions.

Proposition 3.2.4. *Let $\alpha, \beta \in \omega_1$ be distinct ordinals and $k < \Delta(\alpha, \beta)$. Then $\Xi_\delta(\Delta(\alpha, \beta)) = -1$ for each $\delta \in (\alpha)_k \cap (\beta)_k$. In particular $|(\alpha)_k \cap (\beta)_k| \leqslant r_{\Delta(\alpha,\beta)}$.*

Proof. Let $\delta \in (\alpha)_k \cap (\beta)_k$. Suppose towards a contradiction that $\Xi_\delta(\Delta(\alpha, \beta)) \neq -1$. Thus, according to the part (c) of Lemma 2.3.9 and the fact that $\rho(\alpha, \delta), \rho(\beta, \delta) \leqslant k < \Delta(\alpha, \beta)$, we have that $\Xi_\alpha(\Delta(\alpha, \beta)) = \Xi_\delta(\Delta(\alpha, \beta)) = \Xi_\beta(\Delta(\alpha, \beta))$. This is impossible since $\Xi_\alpha(\Delta(\alpha, \beta) \neq \Xi_\beta(\Delta(\alpha, \beta))$ due to the point (d) of the same lemma. ∎

Definition 3.2.5. Let $\alpha, \beta \in \omega_1$ be distinct ordinals and $k = \Delta(\alpha, \beta) - 1$. We define $c(\alpha, \beta)$ as:
$$c_\beta^\alpha = \min(\,(\alpha)_k \backslash (\beta)_k\,)$$

Remark 3.2.6. Note that $c_\beta^\alpha \neq c_\alpha^\beta$ for any two distinct $\alpha, \beta \in \omega_1$. Furthermore, if $k = \Delta(\alpha, \beta) - 1$, then $(c_\beta^\alpha)_k^- = (\alpha)_k \cap (\beta)_k = (c_\alpha^\beta)_k^-$. In this way, $\|c_\alpha^\beta\|_k = \|c_\beta^\alpha\|_k \leqslant r_{k+1}$ due to the Proposition 3.2.4.
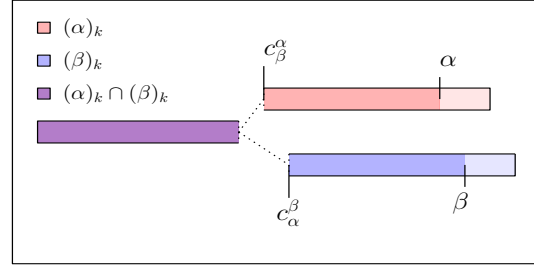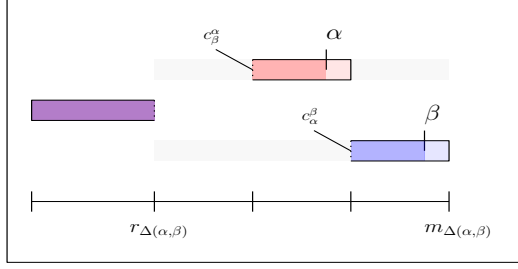

In the next definition we define the relation that turns ω_1 into a Countryman line.
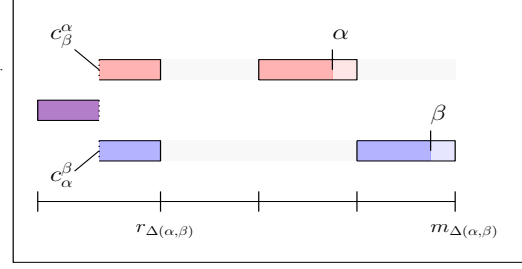
Definition 3.2.7. Let $\mathcal{F}$ be a construction scheme. Given different $\alpha, \beta \in \omega_1$ and $k = \Delta(\alpha, \beta) - 1$, we recursively decide whether $\alpha <_{\mathcal{F}} \beta$ when one of the following conditions holds:

(a) $\|c_\beta^\alpha\|_k = r_{k+1}$ and $\Xi_\alpha(k+1) < \Xi_\beta(k+1)$.

(b) $\|c_\beta^\alpha\|_k < r_{k+1}$ and $c_\beta^\alpha <_{\mathcal{F}} c_\alpha^\beta$.

Where $k = \Delta(\alpha, \beta) - 1$.

In here, $|(\alpha)_k \cap (\beta)_k| = r_{\Delta(\alpha,\beta)}$. Thus, we decide whether $\alpha <_{\mathcal{F}} \beta$ by comparing $\Xi_\alpha(\Delta(\alpha,\beta))$ and $\Xi_\beta(\Delta(\alpha,\beta))$.

In here, $|(\alpha)_k \cap (\beta)_k| < r_{\Delta(\alpha,\beta)}$. Thus, we decide whether $\alpha <_{\mathcal{F}} \beta$ by comparing c_β^α and c_α^β.

Lemma 3.2.8. *Suppose $\alpha, \beta \in \omega_1$ are distinct and let $k = \Delta(\alpha, \beta) - 1$. Consider $h : (\alpha)_k \longrightarrow (\beta)_k$ the increasing bijection and take $\gamma \in (\alpha)_k$ such that $h(\gamma) \neq \gamma$. Then the following statements are equivalent:*

(1) $\alpha <_{\mathcal{F}} \beta$.

(2) $\gamma <_{\mathcal{F}} h(\gamma)$.

Proof. The proof is carried by induction over α and β. So suppose that we have proved the Lemma for each $\alpha' < \alpha$ and $\beta' < \beta$. We start with some remarks about γ. First of all, $\Delta(\gamma, h(\gamma)) \geqslant \Delta(\alpha, \beta)$ due to the point (a) of Lemma 2.3.5. Furthermore, according to Remark 2.3.6 it follows that $\gamma \in (\alpha)_k \backslash (\beta)_k$ and $h(\gamma) \in (\beta)_k \backslash (\alpha)_k$. This means that $c_\beta^\alpha \in (\gamma)_k$ and $c_\alpha^\beta \in (h(\gamma))_k$. Lastly, as $(\alpha)_k \cap (\beta)_k$ is an initial segment of both $(\alpha)_k$ and $(\beta)_k$, we conclude that $(\alpha)_k \cap (\beta)_k = (\delta)_k \cap (h(\delta))_k$. We divide the rest of the proof into two cases.

Case 1: If $\Delta(\alpha, \beta) = \Delta(\gamma, h(\gamma))$.

Proof of case. In this case we have that $c_\beta^\alpha = c_{h(\gamma)}^\gamma$ and $c_\alpha^\beta = c_\gamma^{h(\gamma)}$. Thus, if $\|c_\beta^\alpha\|_k < r_{k+1}$ then $\alpha <_{\mathcal{F}} \beta$ if and only if $c_\beta^\alpha <_{\mathcal{F}} c_\alpha^\beta$ if and only if $\gamma <_{\mathcal{F}} h(\gamma)$. On the other hand, if $\|c_\beta^\alpha\|_k = r_{k+1}$ then $\alpha <_{\mathcal{F}} \beta$ if and only if $\Xi_\alpha(k+1) < \Xi_\beta(k+1)$, and $\gamma <_{\mathcal{F}} h(\gamma)$ if and only if $\Xi_\gamma(k+1) < \Xi_{h(\gamma)}(k+1)$. Now, as $k+1 = \Delta(\gamma, h(\gamma))$ then both $\Xi_\gamma(k+1)$ and $\Xi_{h(\gamma)}(k+1)$ are non-negative. From this fact and the part (c) of Lemma 2.3.9, we conclude that $\Xi_\alpha(k+1) = \Xi_\gamma(k+1)$ and $\Xi_\beta(k+1) = \Xi_{h(\gamma)}(k+1)$. In this way, $\alpha <_{\mathcal{F}} \beta$ if and only if $\gamma <_{\mathcal{F}} h(\gamma)$. $\square$

Case 2: If $\Delta(\alpha, \beta) < \Delta(\gamma, h(\gamma))$.

Proof of case. First we argue that $\alpha <_{\mathcal{F}} \beta$ if and only if $c_\beta^\alpha <_{\mathcal{F}} c_\alpha^\beta$. For this purpose it is enough to show that, in this case, $\|c_\beta^\alpha\|_k < r_{k+1}$. Indeed, according to the point (a) of Lemma 2.3.9, we have that $\Xi_\gamma(k+1) = \Xi_{h(\gamma)}(k+1)$. On the other hand, by the part (d) of such lemma, we also know that $\Xi_\alpha(k+1) \neq \Xi_\beta(k+1)$. Thus, $\Xi_\gamma(k+1) = -1$ due to the part (c) of Lemma 2.3.9. Since $c_\beta^\alpha \in (\gamma)_k$, it follows that $\|c_\beta^\alpha\|_k < r_{k+1}$. In this way, $\alpha <_{\mathcal{F}} \beta$ if and only if $c_\beta^\alpha <_{\mathcal{F}} c_\alpha^\beta$ as we wanted. The proof of this case will be over once we show that $\gamma <_{\mathcal{F}} h(\gamma)$ if and only if $c_\beta^\alpha <_{\mathcal{F}} c_\alpha^\beta$. Let $l = \Delta(\gamma, h(\gamma)) - 1$ and consider

$h' : (\gamma)_l \longrightarrow (h(\gamma))_l$ be the increasing bijection. As $k < l$ then $c_\beta^\alpha \in (\gamma)_k \subseteq (\gamma)_l$. Furthermore,

$$h'(c_\beta^\alpha) = h'(\,(\gamma)_k(\|c_\beta^\alpha\|_k)\,) = (h(\gamma))_k(\|c_\beta^\alpha\|_k) = (h(\gamma))_k(\|c_\alpha^\beta\|_k) = c_\alpha^\beta.$$

As $c_\beta^\alpha \neq c_\alpha^\beta$, we may use the inductive hypotheses to conclude that $\gamma <_{\mathcal{F}} h(\gamma)$ if and only if $c_\beta^\alpha <_{\mathcal{F}} c_\gamma^\beta$. This finishes the proof. $\qquad\square$

$\blacksquare$

Corollary 3.2.9. *let $\alpha, \beta \in \omega_1$ be distinct ordinals. Then $\alpha <_{\mathcal{F}} \beta$ if and only if $c_\beta^\alpha <_{\mathcal{F}} c_\alpha^\beta$.*

Corollary 3.2.10. *Suppose $\alpha, \beta \in \omega_1$ are distinct and let $k < \Delta(\alpha, \beta)$. Consider $h : (\alpha)_k \longrightarrow (\beta)_k$ the increasing bijection and take $\gamma \in (\alpha)_k$ such that $h(\gamma) \neq \gamma$. Then the following statements are equivalent:*

(1) $\alpha <_{\mathcal{F}} \beta$.

(2) $\gamma <_{\mathcal{F}} h(\gamma)$.

Proof. Let $\psi : (\alpha)_{\Delta(\alpha,\beta)-1} \longrightarrow (\beta)_{\Delta(\alpha,\beta)-1}$ be the increasing bijection. To prove the corollary, just notice that $\psi|_{(\alpha)_z} = \phi$. $\qquad\blacksquare$

Lemma 3.2.11. *Let $\alpha, \beta, \delta \in \omega_1$ be distinct ordinals and $k \in \omega$. If $(\alpha)_k \cap (\beta)_k \subseteq (\alpha)_k \cap (\delta)_k$ then $(\alpha)_k \cap (\beta)_k = (\beta)_k \cap (\delta)_k$.*

Proof. Since $(\alpha)_k \cap (\beta)_k \subsetneq (\alpha)_k \cap (\delta)_k$ then $(\alpha)_k \cap (\beta)_k = (\alpha)_k \cap (\delta)_k \cap (\beta)_k \subseteq (\delta)_k \cap (\beta)_k$. In order to prove that $(\beta)_k \cap (\delta)_k \subseteq (\alpha)_k \cap (\beta)_k$, note that both $(\alpha)_k \cap (\delta)_k$ and $(\beta)_k \cap (\delta)_k$ are initial segments of $(\delta)_k$ by virtue of the point (6) in Proposition 2.1.7. In this way, either $(\beta)_k \cap (\delta)_k \sqsubseteq (\alpha)_k \cap (\delta)_k$ or $(\alpha)_k \cap (\delta)_k \sqsubseteq (\beta)_k \cap (\delta)_k$. The latter alternative can not happen as it would imply that $(\alpha)_k \cap (\delta)_k \subseteq (\alpha)_k \cap (\beta)_k$. Thus, the first alternative holds. In this way, $(\beta)_k \cap (\delta)_k \subseteq (\beta)_k \cap (\alpha)_k \cap (\delta)_k \subseteq (\alpha)_k \cap (\beta)_k$. This finishes the proof. $\qquad\blacksquare$

Proposition 3.2.12. *$(\omega_1, <_{\mathcal{F}})$ is a total order.*

Proof. The only non-trivial part of this task is to prove transitivity. This proof will be performed by induction. For this purpose let $\alpha, \beta, \delta \in \omega_1$ be distinct ordinals. Suppose that we have proved that for any $\alpha' < \alpha$, $\beta' < \beta$ and $\delta' < \delta$, the triplet $\{\alpha', \beta', \delta'\}$ do not form a cycle. That is, neither $\alpha' <_{\mathcal{F}} \beta' <_{\mathcal{F}} \delta' <_{\mathcal{F}} \alpha'$ nor $\alpha' >_{\mathcal{F}} \beta' >_{\mathcal{F}} \delta' >_{\mathcal{F}} \alpha'$. We will show that the same holds for $\{\alpha, \beta, \delta\}$. By virtue of Lemma 2.3.4, we may assume without loss of generality that $\Delta(\alpha, \beta) = \Delta(\alpha, \delta) \leqslant \Delta(\beta, \delta)$. Let $k = \Delta(\alpha, \beta) - 1$. For each distinct $x, y \in \{\alpha, \beta, \delta\}$ consider $e_y^x = \min(\,(x)_k \setminus (y)_k\,)$ and $h_y^x : (x)_k \longrightarrow (y)_k$ the increasing bijection. Observe that $h_x^z \circ h_z^y = h_x^y$, $h_y^x(e_y^x) = e_x^y$ and $e_y^x \neq e_x^y$. We divide the rest of the proof into the following cases.

<u>Case 1</u>: There are distinct $x, y, z \in \{\alpha, \beta, \delta\}$ for which $(x)_k \cap (y)_k \subsetneq (x)_k \cap (z)_k$.

Proof of case. By means of the Lemma 3.2.11, $(x)_k \cap (y)_k = (z)_k \cap (y)_k$. It then follows that $e_z^y = e_x^y$. Now we will now show that $e_y^x = e_y^z$. First recall that both $(x)_k \cap (y)_k$ and $(x)_k \cap (z)_k$ are initial segments of $(x)_k$. From this fact we can deduce that $(x)_k \cap (y)_k \subsetneqq (x)_k \cap (z)_k$. By minimality, $e_y^x \in (x)_k \cap (z)_k$. As $e_y^x \notin (y)_k$, it follows that $e_y^x \geqslant e_y^z$. Moreover, since $(x)_k \cap (z)_k \sqsubseteq (z)_k$ then $e_y^z \in (x)_k \cap (z)_k$. Again, we have that $e_y^z \geqslant e_y^x$ because $e_y^z \notin (y)_k$. Therefore, $e_y^z = e_y^x$. According to the Corollary 3.2.10, y and x are ordered (with respect to $<_{\mathcal{F}}$) in the same way as e_x^y and $h_x^y(e_x^y) = e_y^x$, and y and z are ordered in the same way as $e_z^y = e_x^y$ and $h_z^y(e_z^y) = e_y^z = e_y^x$. We conclude that either $y <_{\mathcal{F}} x, z$ or $x, z <_{\mathcal{F}} y$. This finishes the proof of this case. □

<u>Case 2</u>: $(\alpha)_k \cap (\beta)_k = (\alpha)_k \cap (\delta)_k = (\beta)_k \cap (\delta)_k$.

Proof of case. We will divide the proof of this case into two subcases, but first, note that $e_\beta^\alpha, e_\alpha^\beta, e_\delta^\alpha$ and e_α^δ are equal to $c_\beta^\alpha, c_\alpha^\beta, c_\delta^\alpha$ and c_α^δ respectively. Moreover, $e_y^x = e_z^x$ for all distinct $x, y, z \in \{\alpha, \beta, \delta\}$.

<u>Subcase 1</u>: $|(\alpha)_k \cap (\beta)_k| < r_{k+1}$.

Proof of subcase. By the remarks made at the start of this case and by Corollary 3.2.10, we have that α, β and δ are ordered in the same way as e_β^α, e_α^β and e_α^δ respectively. Thus, by virtue of the inductive hypotheses, it suffices to show that $e_\beta^\alpha < \alpha$, $e_\alpha^\beta < \beta$ and $e_\alpha^\delta < \delta$. Indeed, according to the hypotheses of this subcase, $\|e_y^x\|_k = |(x)_k \cap (y)_k| < r_{k+1}$ for all $x, y \in \{\alpha, \beta, \delta\}$. On the other hand, since $k + 1 = \Delta(\alpha, \beta) = \Delta(\alpha, \delta) \leqslant \Delta(\beta, \delta)$, then $\Xi_x(k + 1) \geqslant 0$ for each $x \in \{\alpha, \beta, \delta\}$. In other words $\|x\|_k \geqslant r_{k+1}$. Particularly, $e_\beta^\alpha < \alpha$, $e_\alpha^\beta < \beta$ and $e_\alpha^\delta < \delta$, so we are done. □

<u>Subcase 2</u>: If $|(\alpha)_k \cap (\beta)_k| = r_{k+1}$.

Proof of subcase. According to the definition of the order $<_{\mathcal{F}}$ and since $k + 1 = \Delta(\alpha, \beta) = \Delta(\alpha, \delta)$, it follows that α and β are ordered (with respect to $<_{\mathcal{F}}$) in the same way as $\Xi_\alpha(k + 1)$ and $\Xi_\beta(k + 1)$ (with respect to the usual order), and α and δ are ordered in the same way as $\Xi_\alpha(k + 1)$ and $\Xi_\beta(k + 1)$. If $\Delta(\beta, \delta) > k + 1$ then $\Xi_\beta(k + 1) = \Xi_\delta(k + 1)$ by the point (a) of Lemma 2.3.9. Thus, either $\alpha < \beta, \delta$ or $\beta, \delta < \alpha$. On the other hand, if $\Delta(\beta, \delta) = k + 1$ then $e_\delta^\beta = c_\delta^\beta$ and e_δ^β. Therefore, β and δ are ordered in the same way as $\Xi_\beta(k + 1)$ and $\Xi_\delta(k + 1)$. Evidently, $\Xi_\alpha(k + 1)$, $\Xi_\beta(k + 1)$ and $\Xi_\delta(k + 1)$ do not form a cycle, so we are done. □

There are no more subcases by virtue of the Proposition 3.2.4. Thus, the proof is over. □

∎

Theorem 3.2.13. $(\omega_1, <_{\mathcal{F}})$ *is a Countryman line.*

Proof. We have already seen that $(\omega_1, <_{\mathcal{F}})$ is a total order in Proposition 3.2.12. The only thing left to show is that ω^2 can be partitioned into countably many chains. For

this purpose, it is sufficient to prove the same holds for $D = \{(\alpha, \beta) \in \omega_1^2 \; ; \alpha < \beta\}$. Given $x, y, z \in \omega$, define

$$P(x, y, z) = \{(\alpha, \beta) \in D \; : \; \|\alpha\|_z = x, \; \|\beta\|_z = y \; and \; \rho(\alpha, \beta) = z\}.$$

It is clear that the family of all $P(x, y, z)$ is countable and covers D. Thus, the proof will be over once we show each $P(x, y, z)$ is a chain. Indeed, take $(\alpha, \beta), (\delta, \gamma) \in P(x, y, z)$ and suppose $\alpha <_{\mathcal{F}} \delta$. Since $\|\beta\|_z = \|\gamma\|_z$ then $\Delta(\beta, \gamma) > z$. In this way, we can consider h the increasing bijection from $(\beta)_z$ to $(\gamma)_z$. Note that $\alpha \in (\beta)_z$ and

$$h(\alpha) = h(\,(\beta)_z(\|\alpha\|_z)\,) = h(\,(\beta)_z(x)) = (\gamma)_z(x) = (\gamma)_z(\|\delta\|_z) = \delta.$$

So by Lemma 3.2.8, $\beta <_{\mathcal{F}} \gamma$. ∎

Now, we pass to the study of trees which are closely related to Countryman lines.

Definition 3.2.14 (Aronszajn tree)**.** Let $(T, <)$ be a tree. We say that T is an ω_1-tree if $Ht(T) = \omega_1$, and T_α is countable for each $\alpha < \omega_1$. Furthermore, if T is an ω_1-tree without uncountable chains, we call it an *Aronszajn tree.*

Aronszajn trees were constructed for the first time by Nachman Aronszajn in the 1930's. An important class of Aronszajn trees are the so called *special.* We say that an ω_1-tree, say $(T, <)$, is *special* if it can be written as a countable union of antichains. Since any chain intersect each antichain in at most one point, it follows that each special tree is Aronszajn. It is known that MA implies that any Aronszajn tree is special. On the other hand, the $\Diamond$-principle implies the existence of non-special Aronszajn tree. In the following theorem, we give a simple construction of a special tree from a construction scheme. It is worth pointing out that Aronszajn trees can already be constructed using Countryman lines.

Theorem 3.2.15. *There is a special Aronszajn tree.*

Proof. Let $\mathcal{F}$ be a construction scheme. Given $\beta \in \omega_1$, consider the function $\rho_\beta : \beta + 1 \longrightarrow \omega$ defined as $\rho_\beta(\alpha) = \rho(\alpha, \beta)$. Let

$$T = \{f \in \omega^{<\omega_1} \; : \; \exists \beta \in \omega_1 \, (dom(f) = \beta + 1 \; and \; f =^* \rho_\beta)\}.$$

Given $\alpha < \beta \in \omega_1$, it is straightforward that $\rho_\alpha(\xi) = \rho_\beta(\xi)$ for each $\xi \in (\alpha+1)\backslash(\alpha)_{\rho(\alpha,\beta)}$. From this, we conclude that $f|_{\alpha+1} \in T$ for any $f \in T$ and $\alpha \in dom(f)$. Hence, $(T, \subseteq)$ is an ω_1-tree.

Now, we will show that $(T, \subseteq)$ is in fact a special tree. For this, take an arbitrary $f \in \mathcal{F}$ and let β_f be such that $dom(f) = \beta_f + 1$. Now, consider $k_f \in \omega$ the minimal natural number with the following properties:

(1) $\rho_{\beta_f}(\xi) = f(\xi)$ for each $\xi \notin (\beta_f)_{k_f}$,

(2) $f(\xi) \leqslant k_f$ for each $\xi \in (\beta_f)_{k_f}$.

For each $k, s \in \omega$, let $T(k, s)$ be the set of all $f \in T$ for which $k = k_f$ and $\|\beta_f\|_k = s$.

Claim: $T(k, s)$ is an antichain in T.

Proof of claim. Indeed, take two distinct $f, g \in T(k, s)$. If $\beta_f = \beta_g$ there is nothing to do, so let us assume that $\beta_f < \beta_g$. Since $\|\beta_f\|_k = s = \|\beta_g\|_k$, then $\rho(\beta_f, \beta_g) > k$. By definition of k_f and k_g, we get that $f(\beta_f) \leqslant k < \rho(\beta_f, \beta_g) = g(\beta_f)$. This concludes the claim. $\square$

To finish, just note that $T = \bigcup_{k,s \in \omega} T(k, s)$. $\blacksquare$

3.2.2 Suslin trees

Georg Cantor proved that $\mathbb{R}$ is, up to isomorphism, the unique total order X with the following properties:

(1) X is dense,

(2) X has no end-points,

(3) X is complete. That is, any of its bounded subsets have a supremum,

(4) X is is separable.

In 1920, Mikhail Suslin asked whether the same characterization is true when the property (4) is changed by the condition:

(4)' X is *ccc*. That is, any family of disjoint open intervals in X is at most countable.

Any total order X satisfying the properties (1), (2), (3) and (4)' but not (4) is call a *Suslin line*. Suslin's hypothesis states that there are no Suslin lines. In the 1930's Djuro Kurepa show that the existence of Suslin lines is equivalent to the existence of the trees that we now know as Suslin.

Definition 3.2.16 (Suslin tree)**.** Let $(T, <)$ be an ω_1-tree. We say that T is *Suslin* if $(T, >)$ does not contain uncountable chains or antichains.

Thomas Jech and Stanley Tennenbaum independently proved the consistency of the existence Suslin trees in [55] and [107] respectively. Later, Ronald Jensen showed that the existence of Suslin trees follows from the $\Diamond$-principle. Finally, Robert M. Solovay and Tennenbaum proved that ZFC is consistent with the non existence of Suslin trees (see [100]).

In this subsection, we will show that the existence of two distinct types of Suslin trees follows from the capturing axioms $FCA(part)$ and FCA.

Definition 3.2.17 (Coherent tree)**.** Let $(T, <)$ be a Suslin tree. We say that T is *coherent* if there is a family of functions $\langle f_\alpha \rangle_{\alpha \in \omega_1}$ so that the following properties hold for all $\alpha < \beta \in \omega_1$:

(1) $f_\alpha : \alpha \longrightarrow \omega$,

(2) $f_\alpha =^* f_\beta|_\alpha$.

Furthermore, $T = \{f_\alpha|_\xi : \xi \leqslant \alpha < \omega_1\}$ and the order $<$ of T coincides with $\subseteq$.

Coherent trees have been extensively studied in the past, since they have many interest forcing properties (see [16], [17], [65], [66], [67] and [73]).

Theorem 3.2.18 (Under FCA(part)). *There is a coherent Suslin tree.*

Proof. Let $\tau = \langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$ be a type so that $n_{k+1} \geqslant 2^{m_k - r_{k+1}} + 1$ for each $k \in \omega$. Furthermore, let $\mathcal{P} = \{P_c, P_a\}$ be a partition of ω compatible with τ. We will build a coherent Suslin tree by using a $\mathcal{P}$-fully capturing construction scheme $\mathcal{F}$ of type τ.

For each $k \in \omega$, we first enumerate (possibly with repetitions) the set of all functions from $m_k \backslash r_{k+1}$ into 2 as $\langle g_i^k \rangle_{0 < i < n_{k+1}}$. Now, given $\beta \in \omega_1$, we define $f_\beta : \beta \longrightarrow 2$ as follows:

$$f_\beta(\xi) = \begin{cases} 1 & \text{if } \Xi_\xi(l) = 0, \ \Xi_\beta(l) = 1 \text{ and } l \in P_c \\ g_i^l(\|\xi\|_l) & \text{if } \Xi_\xi(l) = 0 \text{ and } l \in P_a \\ 0 & \text{otherwise} \end{cases}$$

Here, $l = \rho(\xi, \beta)$.

Before defining the tree, we will prove that the sequence $\langle f_\alpha \rangle_{\alpha \in \omega_1}$ satisfies the condition (2) of the Definition 3.2.17. This will be done in the following claim.

<u>Claim 1</u>: Let $\xi < \alpha < \beta \in \omega_1$. If $\xi \notin (\alpha)_{\rho(\alpha,\beta)}$ then $f_\alpha(\xi) = f_\beta(x)$.

Proof of claim. Note that $\rho(\xi, \alpha) > \rho(\alpha, \beta)$ we the hypotheses. As ρ is an ordinal metric, we use the previous fact to conclude that $\rho(\xi, \alpha) = \rho(\xi, \beta)$. Let us call this number l. According to the part (c) of Lemma 2.3.9, either $\Xi_\alpha(l) = -1$ or $\Xi_\alpha(l) = \Xi_\beta(l)$. On the other hand, $0 \leqslant \Xi_\xi(l) < \Xi_\alpha(l)$ due to part (b) of the same lemma. In this way, $\Xi_\alpha(l) = \Xi_\beta(l)$. By definition of f_α and f_β, $f_\beta(\xi) = f_\alpha(\xi)$. $\square$

Now we define T as expected. That is, $T = \{f_\alpha|_\xi : \xi \leqslant \alpha < \omega_1\}$ and consider it ordered by $\subseteq$. In the next two claims, we will prove that T is the coherent Suslin tree we are looking for.

<u>Claim 2</u>: T does not have uncountable chains.

Proof of claim. Let $S \in [T]^{\omega_1}$. Without loss of generality we can suppose that each element of S is of the form $f_\alpha|_\xi$ for some $\xi < \alpha \in \omega_1$. Consider

$$\mathcal{C} = \{C \in [\omega_1]^2 : f_{C(1)}|_{C(0)} \in S\}.$$

By refining $\mathcal{C}$, we may assume that its elements are pairwise disjoint. Since $\mathcal{F}$ is $\mathcal{P}$-3-capturing, there are distinct $C_0, C_1, C_2 \in \mathcal{C}$ so that $\{C_0, C_1, C_2\}$ is captured at some level $l \in P_c$. For convenience, let us denote $C_i(0)$ and $C_i(1)$ simply as ξ_i and α_i for each $i < 3$. In order to finish, just note that $\Xi_{\alpha_1}(l) = 1$, $\Xi_{\alpha_2}(l) = 2$ and $\rho(\xi_0, \alpha_1) =$

$\rho(\xi_0, \alpha_2) = l$. Thus, according to the definition, we have that $f_{\alpha_1}(\xi_0) = f_{\alpha_1}|_{\xi_1}(\xi_0) = 1$ and $f_{\alpha_2}(\xi_0) = f_{\alpha_2}|_{\xi_2}(\xi_0) = 0$. Hence, S is not a chain. $\square$

<u>Claim 3</u>: T has no uncountable antichains.

Proof of claim. Let $A \in [T]^{\omega_1}$. Without loss of generality we can suppose that each element of A is of the form f_α for some $\alpha \in \omega_1$. Consider

$$\mathcal{D} = \{\alpha \in \omega_1 \,:\, f_\alpha \in A\}.$$

Since $\mathcal{F}$ is $\mathcal{P}$-fully capturing, there is $D \in \mathrm{FIN}(\mathcal{D})$ which is fully-captured at some level $l \in P_a$. For convenience, let us denote $D(i)$ as α_i for each $i < l$. Now, let $g : m_{l-1}\backslash r_l \longrightarrow 2$ be given by:

$$g(j) = \begin{cases} f_{\alpha_0}((\alpha_0)_l(j)) & \text{if } j < \|\alpha_0\|_l \\ 0 & \text{otherwise} \end{cases}$$

Take $0 < i < l$ for which $g = g_i^l$. We claim that $f|_{\alpha_0} = f_{\alpha_i}|_{\alpha_0}$. For this purpose, take an arbitrary $\xi < \alpha_0$. If $\xi \in (\alpha_0)_l$ and $r_l \leqslant \|\xi\|_l$ then

$$f_{\alpha_i}(\xi) = g_i^l(\|\xi\|_l) = g(\|\xi\|_l) = f_{\alpha_0}((\alpha_0)_l(\|\xi\|_l)) = f_{\alpha_0}(\xi).$$

If $\xi \in (\alpha_0)_l$ and $\|\xi\|_l < r_l$ then $\xi \in (\alpha_0)_l \cap (\alpha_i)_l$. By means of the Proposition 2.4.10, we know that $\rho(\alpha_0, \alpha_i) = \Delta(\alpha_0, \alpha_i) = l$. In other words, α_0 and α_i are strongly ρ-isomorphic (see Remark 2.4.7). From this, it follows that $\rho(\xi, \alpha_0) = \rho(\xi, \alpha_i)$. Thus, $f_{\alpha_i}(\xi) = f_{\alpha_0}(\xi)$ by virtue of the part (a) of Lemma 2.3.9 (since $\Delta(\alpha_0, \alpha_i) = l$). The last case happen when $\xi \notin (\alpha)_l$. Here, $f_{\alpha_0}(\xi) = f_{\alpha_i}(\xi)$ as a direct consequence of the Claim 1. $\square$

$\blacksquare$

Definition 3.2.19 (Tree product)**.** Given $k \in \omega$ and $\{(T_i, <_i)\}_{i<k}$ a family of trees we define their *tree product*

$$\bigotimes_{i<k} T_i = \{t \in \prod_{i<k} T_i \,:\, \forall i < k \left(rank(t(0)) = rank(t(i))\right)\}.$$

To this set, we associate a canonical order given by:

$$s < t \text{ if and only if } s(i) <_i t(i) \text{ for each } i < k$$

Remark 3.2.20. It is not hard to see that $\bigotimes\limits_{i<k} T_i$ is always a tree. Furthermore, the tree product of Aronszajn trees is always Aronszajn. Unfortunately, we can not say the same about the tree product of Suslin trees. In fact, the tree product of a Suslin tree with itself is never Suslin.

Definition 3.2.21 (Full Suslin tree)**.** Let $(T, <)$ be a tree. We say that T is *full Suslin* if $\bigotimes\limits_{i<k} T|_{t_i}$ is Suslin for every distinct $t_0, \ldots, t_{k-1} \in T$, all of the same rank.

Theorem 3.2.22 (Under FCA). *There is a full Suslin tree.*

Proof. Fix a type $\langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$ such that $n_{k+1} \geqslant 2^{m_k}$ for each $k \in \omega$, and let $\mathcal{F}$ be a fully capturing construction scheme of that type. The plan is define an order $<_T$ over ω_1 which turns it into a full Suslin tree. For this purpose, we will recursively define for each $F \in \mathcal{F}$, a tree ordering $\prec_F$. We will ask that the following conditions hold for any two $F, G \in \mathcal{F}$:

1. For each $\alpha, \beta \in F$, if $\alpha \prec_F \beta$ then $\alpha < \beta$ in the usual ordering over ω_1.

2. If $F \subseteq G$, then $\prec_F = \prec_G \cap (F \times F)$.

3. If $\rho^F = \rho^G$ and $h : F \longrightarrow G$ is the increasing bijection, then h is an isomorphism between $(F, \prec_F)$ and $(G, \prec_G)$.

We proceed to define the orderings by recursion over $k = \rho^F$.

<u>Base step</u>: If $k = 0$, there is nothing to do. This is because $|F| = 1$ in this case.

<u>Recursion step</u>: Suppose that $0 < k \in \omega$ and we have constructed the required orderings over each element of $\mathcal{F}_{k-1}$. Let $F \in \mathcal{F}_k$. For each $i < n_k$ let $h_i : F_0 \longrightarrow F_i$ be the increasing bijection.

Given $I \subseteq F_0 \backslash R(F)$, let us say that I is *k-independent* if for any two distinct $\alpha, \beta \in I$ and each $\xi \in F_0$, if $\xi \prec_{F_0} \alpha, \beta$ then $\xi \in R(F)$. Note that both the empty set and singletons are *k*-independent. Now, let us enumerate (possibly with repetitions) the set of all *k*-independent sets of $F_0 \backslash R(F)$ as $\langle I_i^F \rangle_{0 < i < n_k}$. Given $\alpha, \beta \in F$, we decide whether $\alpha \prec_F \beta$ if one of the two following cases occur:

- Both α and β belong to the same F_i and $\alpha \prec_{F_i} \beta$.

- $\alpha \in F_0 \backslash R(F)$, $\beta \in F_i \backslash R(F)$ for some $i > 0$ and there is a (necessarily unique) $\delta \in I_i^F$ so that $\alpha \preceq_{F_0} \delta$ and $h(\delta)$ is comparable with β.

It is straightforward to check that $\prec_F$ is a tree ordering which satisfies the condition (2) of the recursive hypotheses. Furthermore, in the definition we are only adding new relations between elements of $F_0 \backslash R(F)$ and $F_i \backslash R(F)$ for $i > 0$, it also follows that the condition (1) of the recursive hypotheses hold. It should be also clear that condition (3) holds if we choose the enumerations $\langle I_i^F \rangle_{0 < i < n_k}$ always in "isomorphic" positions. This finishes the construction.

We now define $\prec = \bigcup_{F \in \mathcal{F}} \prec_F$. According to the points (1), (2) and (3) of the recursion hypotheses, it follows that $(\omega_1, <_T)$ is in fact a tree. Moreover, $\prec \cap (F \times F) = \prec_F$ for any $F \in \mathcal{F}$. We proceed to show that it is full Suslin.

<u>Claim 1</u>: $(\omega_1, <_T)$ does not have uncountable chains.

Proof of claim. Let $A \in [\omega_1]^{\omega_1}$. As $\mathcal{F}$ is full capturing there is $l \in \omega$ and $D \in [A]^l$ which is captured at level l. Consider $F \in \mathcal{F}_l$ so that $D \subseteq F$. $I = \emptyset$ is l-independent. Hence, there is $0 < i < n_l$ so that $I = I_i^F$. By definition of $\prec_F$, we have that $D(0)$ is incompatible with $D(1)$. So the claim is over. $\square$

<u>Claim 2</u>: Let $1 \leqslant k \in \omega$ and $t_0 < \ldots < t_{k-1} \in \omega_1$ be distinct elements of the same rank. Then $\bigotimes_{i<k} \omega_1|_{t_i}$ has no uncountable antichains.

Proof of claim. Let $\mathcal{A}$ be an uncountable subset of $\bigotimes_{i<k} \omega_1|_{t_i}$. We may assume without loss of generality that $t_{k-1} < s(0) < \ldots < s(k-1)$ for any $s \in \mathcal{A}$. Furthermore, we may suppose that if s and t are distinct elements of $\mathcal{A}$, either $s(k-1) < t(0)$ or $t(k-1) < s(0)$. Given $s \in \mathcal{A}$, let

$$D_s = \{t_0, \ldots, t_{k-1}, s(0), \ldots, s(k-1)\}.$$

As $\mathcal{F}$ is fully capturing, there $s_0, \ldots, s_{l-1} \in \mathcal{A}$ so that $\mathcal{D} = \{D_{s_0}, \ldots, D_{s_{n_{l-1}}}\}$ is fully captured at some level $l \in \omega$. Let $F \in \mathcal{F}_l$ for which

$$\bigcup D_{s_i} \subseteq F.$$

Then $D_{s_i} \subseteq F_i$ for any $i < n_l$. Moreover, $F_i \cap R(F) = \{t_0, \ldots, t_{k-1}\}$. We will show that there is $i < n_l$ so that $s_0 < s_i$ with respect to the tree product ordering. For this aim, we need the following subclaim:

<u>Subclaim 1</u>: $I = \{s_0(0), \ldots, s_1(k-1)\}$ is an l-independent subset of $F_0 \backslash R(F)$.

Proof of subclaim. Let $i < j < k$ and $\alpha \in F_0$ be such that $\alpha \prec s_0(i), s_0(j)$. We know that $t_i \prec s_0(i)$ and $t_j \prec s_0(j)$. Furthermore, t_i and t_j are incomparable because they are different elements of the same rank. Hence, it must happen that $\alpha \prec t_i, t_j$. By the point (1) of the recursive conditions, this implies that $\alpha < t_i, t_j$. As $t_i \in R(F)$, then $\alpha \in R(F)$ too. This completes the proof. $\square$

By virtue of the previous subclaim, there is $i < n_l$ for which $I_i^F = I$. Let us consider $h_i : F_0 \longrightarrow F_i$ the increasing bijection. Given $j < k$, we have that $s_0(j) \prec s_i(j)$ because there is $\delta \in I_i^F$, namely $s_0(j)$ so that $s_0(j) \prec \delta$ and $h(\delta) = s_i(j)$ is comparable with $s_i(j)$. In this way, we conclude that $s_0 < s_j$ with respect to the tree product ordering. This finishes the proof. $\square$

$\blacksquare$

We will finish this subsection by showing that 2-capturing schemes do not suffice to construct Suslin trees. For this task, we will need the following Lemma.

Lemma 3.2.23. *Let $\mathcal{F}$ be a 2-capturing construction scheme. Given $X \in [\omega_1]^{\omega_1}$, the set $\{\alpha \in X : \{\beta \in X : \{\alpha, \beta\}$ is captured$\}$ is uncountable$\}$ is co-countable in X.*

Proof. Let X be as in the hypotheses. Suppose towards a contradiction that there are uncountably many $\alpha \in X$ for which the set $C_\alpha = \{\beta \in X : \{\alpha, \beta\} \text{ is captured }\}$. Then we can recursively construct an uncountable $Y \subseteq X$ so that if $\alpha < \beta \in Y$, then $\beta > \sup(C_\alpha \cup \{0\})$. As $\mathcal{F}$ is assumed to be 2-capturing. There are $\alpha, \beta \in Y$ for which $\{\alpha, \beta\}$ is captured. Since $Y \subseteq X$, then $\beta \in C_\alpha$ which is a contradiction.

∎

Theorem 3.2.24. *Let $\mathcal{F}$ be a 2-capturing construction scheme. If $\mathfrak{m}_\mathcal{F} > \omega_1$, then there are no Suslin trees.*

Proof. Let us assume towards a contradiction that there is a Suslin tree T. It is well known that Suslin trees are never Knaster. Therefore, by Propositions 4.1.3 and 4.0.3, there are $X \in [T]^{\omega_1}$ and an injective function $\zeta : X \longrightarrow \omega_1$ in such way that for any two distinct $x, y \in X$, if $\{\zeta(x), \zeta(y)\}$ is captured, then x and y are incompatible. We now define

$$\mathbb{P} = \{p \in [X]^{<\omega} : p \text{ is an antichain in } T\}$$

and order it with the reverse inclusion.

<u>Claim</u>: $\mathbb{P}$ is *ccc* and 2-preserves $\mathcal{F}$.

Proof of claim. We will prove the claim by appealing to the Lemma 4.0.3. Let $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$ and $\nu : \mathcal{A} \longrightarrow \omega_1$ be an injective function. Given $p \in \mathcal{A}$, let us define

$$A_p = \zeta[p] \cup \{\nu(p)\},$$

$$\alpha_p = \max(A_p).$$

Without any loss of generality we may assume that the elements of $\{A_p : p \in \mathcal{A}\}$ are pairwise disjoint. Furthermore, we can suppose that there are $n, k, a \in \omega$ so that the following conditions hold for any two distinct $p, q \in \mathcal{A}$:

(a) $|p| = n$.

(b) $\rho^{A_p} = k$.

(c) $\|\alpha_p\|_k = a$.

(d) $|A_p| = |A_q|$. Furthermore, if $h : (\alpha_p)_k \longrightarrow (\alpha_q)_k$ is the increasing bijection, then $h[A_p] = A_q$ and $h(\nu(p)) = \nu(q)$.

(e) $\rho(\alpha_p, \alpha_q) > k$.

We now enumerate each $p \in \mathcal{A}$ as $x_0^p, \dots, x_{n-1}^p$ in such way that $\zeta(x_i^p) < \zeta(x_j^p)$ whenever $i < j$. By virtue of the Lemma 2.4.11 and the points (b), (c), (d) and (e) above, if $p, q \in \mathcal{A}$ are such that $\{\alpha_p, \alpha_q\}$ is captured, then so are $\{\nu(p), \nu(q)\}$ and $\{\zeta(x_i^p), \zeta(x_i^q)\}$. The claim follows directly from the next subclaim.

<u>Subclaim</u>: There are distinct $p, q \in \mathcal{A}$ so that $p \cup q$ is an antichain and $\{\alpha_p, \alpha_q\}$ are captured.

Proof of subclaim. Let us suppose that the subclaim is false. Then, given $p, q \in \mathcal{A}$, if $\{\alpha_p, \alpha_q\}$ is captured, then $p \cup q$ is not an antichain. In this way, there are $i_{p,q}, j_{p,q} < n$ for which $x^p_{i_{p,q}}$ and $x^q_{j_{p,q}}$ are different but comparable in T. By means of Lemma 3.2.23, we can construct three sequences $\langle p_s \rangle_{s < n^2+1} \subseteq \mathcal{A}$, $\langle (i_s, j_s) \rangle_{s < n^2+1} \subseteq n \times n$ and $\langle M_s \rangle_{s < n^2+1} \subseteq [\mathcal{A}]^{\omega_1}$ so that the following properties are satisfied for each $s < n^2 + 1$:

- For all $q \in M_s$, $\{\alpha_{p_s}, \alpha_q\}$ is captured and $(i_s, j_s) = (i_{p_s,q}, j_{p_s,q})$.

- $p_{s+1} \in M_s$.

- $M_{s+1} \subseteq M_s$.

By the pigenhole principle, there are $s < r < n^2 + 1$ for which $(i_s, j_s) = (i_r, j_r) = (i, j)$ for some $i, j < n$. Since M_r is uncountable but $x^{p_s}_i {\downarrow} \cup x^{p_r}_i {\downarrow}$ is countable, there is $q \in M_r$ for which $x^{p_s}_i, x^{p_r}_i < x^q_j$. As T is a tree, then $x^{p_s}_i$ and $x^{p_r}_i$ are comparable. On the other hand, $p_r \in M_s$. Thus, $\{\alpha_{p_s}, \alpha_{p_r}\}$ is captured. So by the observations prior to the subclaim, this means that $\{\zeta(x^{p_s}_i), \zeta(x^{p_r}_i)\}$ is captured too. Therefore, $x^{p_s}_i$ and $x^{q_s}_i$ are incompatible. This is a contradiction, so the proof is over. $\square$

$\square$

$\mathbb{P}$ is an uncountable *ccc* forcing which 2-preserves $\mathcal{F}$. As $\mathfrak{m}_{\mathcal{F}} > \omega_1$, such forcing contains an uncountable filter, namely G. Note that $\bigcup G$ is an uncountable chain in T. This contradicts the fact that T is Suslin. ∎

3.2.3 Suslin lattices

Recall that a pie in a partial order X is a set of pairwise incomparable elements.

Definition 3.2.25 (Suslin lattice)**.** Let $(X, <, \wedge)$ be a lower semi-lattice. We say that $(X, <, \wedge)$ is *Suslin* if

- $(X, <)$ is well founded,

- X is uncountable,

- X does not contain any uncountable chain nor an uncountable pie.

Suslin lower semi-lattices were first studied by Stephen J. Dilworth, Edward Odell and Bünyamin Sari in [28], in the context of Banach spaces. In [86], Dilip Raghavan and Teruyuki Yorioka proved that, assuming the $\diamondsuit$-principle, there is a Suslin lower semi-lattice $\mathbb{S}$ which is a substructure of $(\mathscr{P}(\omega), \subseteq, \cap)$ and such that $\mathbb{S}^n$ does not contain any uncountable pie for each $n \in \omega$. A partial order which satisfies this last property is said to be a *powerful pie*.
In the following theorem, we show that CA_2 is all that is needed in order to construct a Suslin lower semi-lattice with the above mentioned properties.

Theorem 3.2.26 (Under CA_2)**.** *There is $\mathbb{S} \subseteq \mathscr{P}(\omega)$ such that $(\mathbb{S}, \subseteq, \cap)$ is a Suslin lower semi-lattice which is powerful pie.*

Proof. Let $\mathcal{F}$ be a 2-capturing morass. For each $k \in \omega$, let $A_k = m_k \times 2^k$. We the $U_k = \{k\} \times m_k \times 2^{k-1}$. Since the last expression has no sense when $k = 0$, we let $U_0 = \{(0,0,0)\}$. Also, let $\phi_k : A_k \longrightarrow A_{k+1}$ be given as:

$$\phi_k(a,b) = \begin{cases} (a,b) & \text{if } (a,b) \in r_{k+1} \times 2^k \\ (a + (m_k - r_{k+1}), b) & \text{in other case} \end{cases}$$

Notice that $A_k \cup \phi_k[A_k] = m_{k+1} \times 2^k$. As the final part of the preparation, let

$$N_k = \bigcup_{i \leqslant k} U_k.$$

Our first objective is to construct, for each $k \in \omega$, a family $\langle S_x^k \rangle_{x \in A_k} \subseteq \mathscr{P}(N_k)$ in such way that the following conditions hold for $\mathbb{S}^k = \{\emptyset\} \cup \{S_x^k : x \in A_k\}$:

(a) $(\mathbb{S}^k, \subseteq, \cap)$ is a lower semi-lattice.

(b) $\mathbb{S}_0^k = \{\emptyset\}$ and $\mathbb{S}_{i+1}^k = \langle S_x^k \rangle_{x \in \{i\} \times 2^k}$ for all $i < m_k$ (where $\mathbb{S}_i^k$ is the set of all elements of $\mathbb{S}^k$ of rank i).

(c) For all $x \in A_k$, $S_x^{k+1} = S_x^k$ and $S_{\phi_k(x)}^{k+1} \cap N_k = S_x^k$. In particular, $S_x^k \subseteq S_{\phi_k(x)}^{k+1}$.

(d) The function $\psi_k : \mathbb{S}^k \longrightarrow \mathbb{S}^{k+1}$ given as:

$$\psi_k(x) = \begin{cases} \emptyset & \text{if } x = \emptyset \\ S_{\phi_k(y)}^{k+1} & \text{if } x = S_y^k \end{cases}$$

is an lower semi-lattice embedding for each $k \in \omega$.

The construction is carried by recursion over k.

Base step: If $k = 0$, then $A_k = \{(0,0)\}$. In this case, we let $S_{(0,0)}^0 = \{(0,0,0)\}$. Trivially, all the conditions are satisfied.

Recursive step: Suppose that we have defined $\mathbb{S}^k$ for some $k \in \omega$ in such way that the conditions (a), (b), (c) and (d) are satisfied. In order to define $\mathbb{S}^{k+1}$, we first divide A_{k+1} into three quadrants:

■ $C_0 = A_k$,

■ $C_1 = [m_k, m_{k+1}) \times 2^k$,

■ $C_2 = m_{k+1} \times [2^k, 2^{k+1})$,

Now, take an arbitrary $x \in A_{k+1}$ and consider the following cases:

(i) If $x \in C_0$, let $S_x^{k+1} = S_x^k$.

(ii) If $x \in C_2$, then $x = (a,b)$ for some $a < m_{k+1}$ and $2^k \leqslant b < 2^{k+1}$. In this case, let $S_x^{k+1} = \{k+1\} \times (a+1) \times \{b - 2^k\}$.

(iii) If $x \in C_1$, let $z = \phi_k^{-1}(x)$ and consider $D_x = \{b < 2^k : S_{(r_{k+1},b)}^k \subseteq S_z^k\}$. Observe that D_x codes the elements of $\mathbb{S}_{r_k+1}^k$ which are below S_z^k. In this case, let

$$S_x^{k+1} = S_z^k \cup \Big(\bigcup_{b \in D_x} \{k+1\} \times m_k \times \{b\} \Big).$$

It follows directly that condition (c) is satisfied. In particular, $\mathbb{S}^k \subseteq \mathbb{S}^{k+1}$. We will prove that $\mathbb{S}^{k+1}$ is a lower semi-lattice by showing that it is closed under intersections. For this purpose, let $x, y \in A_{k+1}$ and consider the following cases:

<u>Case 1</u>: If $x, y \in C_0$.

Proof of case. In this case, $S_x^{k+1}, S_y^{k+1} \in \mathbb{S}^k$, Thus, $S_x^{k+1} \cap S_y^{k+1} \in \mathbb{S}^k \subseteq \mathbb{S}^{k+1}$ due to the recursive hypotheses. $\square$

<u>Case 2</u>: If $x, y \in C_1$.

Proof of case. Let $x' = \phi_k^{-1}(x)$ and $y' = \phi_k^{-1}(y)$. If $D_x \cap D_y = \emptyset$, thdn $S_x^{k+1} \cap S_y^{k+1} = S_{x'}^k \cap S_{y'}^k \in \mathbb{S}^k$. On the other hand, if the intersection of $D_x \cap D_y \neq \emptyset$, then $S_{x'}^k \cap S_{y'}^k = S_w^k$ for some $w \in [r_{k+1}, m_k) \times 2^k$. As $\mathbb{S}^k$ satisfies the conditions (a) and (b), we conclude that $D_x \cap D_y = D_{\phi_k(w)}$. Thus, $S_x^{k+1} \cap S_y^{k+1} = S_{\phi_k(w)}^{k+1}$. $\square$

<u>Case 3</u>: If $x, y \in C_2$.

Proof of case. We can suppose without loss of generality that the first coordinate of x is smaller or equal than the first coordinate of y. In this case, $S_x^{k+1} \cap S_y^{k+1} = S_x^{k+1}$ if x and y share the second coordinate, or $S_x^{k+1} \cap S_y^{k+1} = \emptyset$ if their second coordinates are distinct. $\square$

<u>Case 4</u>: If $x \in C_0$ and $y \in C_1$.

Proof of case. By definition we have that $S_x^{k+1} \cap S_y^{k+1} = S_x^k \cap S_{\phi^{-1}(y)}^k$. $\square$

<u>Case 5</u>: If $x \in C_0$ and $y \in C_2$.

Proof of case. In here, $S_x^{k+1} \cap S_y^{k+1} = \emptyset$. $\square$

<u>Case 6</u>: If $x \in C_1$ and $y \in C_2$.

Proof of case. let $a < m_{k+1}$ and $2^k \leqslant b < 2^{k+1}$ such that $y = (a, b)$. If $b - 2^k \in D_x$, we have $S_x^{k+1} \cap S_y^{k+1} = S_{(c,b)}^{k+1}$ where $c = \min(a, r_{k+1})$. On the other hand, if $b - 2^k \notin D_x$ then $S_x^{k+1} \cap S_y^{k+1} = \emptyset$. $\square$

In this way, we finish the proof of the satisfaction of condition (a). By carefully looking at the equalities in cases 1, 2 and 3, we also conclude that condition (d) is true for $\mathbb{S}^{k+1}$.

We now proceed to check condition (b). For this, first observe that for each $(a, b) \in C_0 \cup C_2$ it is trivially true that $rank(S_{(a,b)}^{k+1}) = a + 1$. To prove the same holds for all $x = (a, b) \in C_1$, we use induction over the first coordinate. The base case is when $a = m_k$. Here, we have that $S_x^{k+1} = S_{(r_{k+1},b)}^{k+1} \cup S_{(m_k-1,b+2^k)}^{k+1}$. Furthermore, any element

of $\mathbb{S}^{k+1}$ which is contained in S_x^{k+1}, is also contained in either $S_{(r_k,b)}^{k+1}$ or $S_{(m_k-1,b+2^k)}^{k+1}$. This means

$$rank(S_{(m_k,b)}^{k+1}) = \max(r_{k+1}+1, m_k+1) = m_k + 1.$$

For the inductive step, suppose we that have proved what we want for each $(c,d) \in C_1$ with $c < a$. Let L be the set of all $(a-1,d)$ for which $S_{(a-1,d)}^{k+1}$ is (properly) contained in S_x^{k+1}. By the inductive hypotheses, $rank(S_y^{k+1}) = a$ for each $y \in L$. The key now is to notice that each element of $\mathbb{S}^{k+1}$ which is properly contained in S_x^{k+1} is also contained in S_y^{k+1} for some $y \in L$. $rank(S_x^{k+1}) = a+1$ as a direct consequence of this fact.

Given $k \in \omega$, let us define $f_k : \omega_1 \times \omega \longrightarrow \omega^2$ by the formula $f_k(\alpha, b) = (\|\alpha\|_k, b)$. For each $(\alpha, b) \in \omega_1 \times \omega$, let

$$S_{(\alpha,b)} = \bigcup_{k>b} S_{f_k(\alpha,b)}^k.$$

Finally, let $\mathbb{S} = \{\emptyset\} \cup \{S_x : x \in \omega_1 \times \omega\}$. We will prove that $\mathbb{S}$ is a Suslin lower semi-lattice. As before, it suffices to show that it is closed under intersections. Indeed, take $x = (\beta, b), y = (\delta, d) \in \omega_1 \times \omega$. According to the conditions (c) and (d), $S_x \cap N_k = S_{f_k(x)}^k$ and $S_y \cap N_k = S_{f_k(y)}^k$ for each $k > \max(b, d)$. Observe that $f_k(x), f_k(y) \in A_k$ where $k = \max(\rho(\beta, \delta), b, d)$. By condition (a), we know that $S_{f_k(x)}^k \cap S_{f_k(y)}^k \in \mathbb{S}^k$. If this intersection is empty, we can use condition (d) to conclude that $S_x \cap S_y = \emptyset$. On the other hand, if $S_{f_k(x)}^k \cap S_{f_k(y)}^k = S_{(a,c)}^k$ for $(a,b) \in A_k$, then $a < \|\beta\|_k$. Thus, there is $\alpha \in (\beta)_k$ for which $\|\alpha\|_k = a$. In this way, $S_{f_k(x)}^k \cap S_{f_k(y)}^k = S_{f_k(\alpha,c)}^k$. From conditions (c) and (d) we conclude that $S_x \cap S_y = S_{(\alpha,c)}$. Now, the condition (b) implies $\mathbb{S}$ is well-founded and its rank function satisfies the following properties:

- $rank(\emptyset) = 0$,

- $rank(\alpha, b) = \alpha + 1$ if $\alpha \in \omega$,

- $rank(\alpha, b) = \alpha$ if $\alpha \in \omega_1 \backslash \omega$.

We do not have to prove that $\mathbb{S}$ has no uncountable chains since it is a substructure of $(\mathscr{P}(N), \subseteq, \cap)$, where $N = \bigcup_{i \in \omega} N_i$. Thus, the only thing left to do is to prove $\mathbb{S}$ is powerful pie. For this, let $n \in \omega$ and $\mathcal{A}$ be an uncountable subset of $(\omega_1 \times \omega)^n$. For each $x \in \mathcal{A}$, let (α_i^x, b_i^x) be such that $x(i) = (\alpha_i^x, b_i^x)$ for all $i < n$. Without any loss of generality we may assume that the following conditions hold for any two distinct $x, y \in \mathcal{A}$:

- $\alpha_0^x < \ldots < \alpha_{n-1}^x$.

- $\{\alpha_j^x : j < n\} \cap \{\alpha_j^y : j < n\} = \emptyset$

We also suppose can suppose there are $b_0, \ldots, b_{n-1} \in \omega$ such that $b_i^x = b_i$ for each $i < n$ and $x \in \mathcal{A}$. Since $\mathcal{F}$ is 2-capturing, there are distinct $x, y \in \mathcal{A}$ for which $\{\langle \alpha_i^x \rangle_{i<n}, \langle \alpha_i^y \rangle_{i<n}\}$ is captured at some level $l > \max(b_i : i < n) + 1$. Note that

$$f_{l-1}(x(i)), f_{l-1}(y(i)) \in A_{l-1}$$

for each $i < n$. Now, given $i < n$ we have that $\rho(\alpha_i^x, \alpha_i^y) = \Delta(\alpha_i^x, \alpha_i^y) = l$. From this, it follows that $\|\alpha_i^x\|_{l-1} = \|\alpha_i^x\|_l$ and $\|\alpha_i^y\|_l = \|\alpha_i^x\|_{l-1} + (m_{l-1} - r_l)$. Thus, $f_{l-1}(x(i)) = f_l(x(i))$ and

$$\phi_{l-1}(f_{l-1}(x(i))) = (\|\alpha_i^x\|_{l-1} + (m_{l-1} - r_l), b_i) = (\|\alpha_i^y\|_l, b_i) = f_l(y(i)).$$

By condition (c), we conclude that $S_{f_l(x(i))}^l = S_{f_{l-1}(x(i))}^{l-1} = S_{f_l(y(i))}^l \cap N_k \subseteq S_{f_l(y(i))}^l$. By a previous argument, this means $S_{x(i)} \subseteq S_{y(i)}$ for each $i < n$. Consequently, x and y testify that $\{(S_{z(0)}, \ldots, S_{z(n-1)}) : z \in \mathcal{A}\}$ is not a pie. This finishes the proof. ∎

3.2.4 Entangled sets

One of Georg Cantor's most famous theorems is that any two countable dense total orders without endpoint are isomorphic. Particularly, this result can be applied to any two countable dense subsets of the reals. Thus, it is natural to ask whether Cantor's theorem can be extended to higher infinities (inside $\mathbb{R}$).

Definition 3.2.27 (κ-dense sets)**.** Let κ be an infinite cardinal. We say that $D \subseteq \mathbb{R}$ is κ-*dense* if $|(a, b) \cap D| = \kappa$ for all $a < b \in D$.

In [9], James Baumgartner proved that it is consistent that any two ω_1-dense sets of reals are isomorphic. This assertion is now known as the Baumgartner's axiom for ω_1. More generally, if κ is an infinite cardinal then we can define:

Baumgartner's Axiom for κ $[BA(\kappa)]$**:** Any two κ-dense sets of reals are isomorphic.

One of the ways of proving Cantor's theorem is by defining the forcing of finite approximations of isomorphisms and making use of the Rasiowa Sikorski's Lemma. Hence, it is natural to think that maybe $BA(\kappa)$ has some relation with MA. However, this is not the case. In [7], Uri Abraham and Saharon Shelah showed that MA does not imply $BA(\omega_1)$. For this, they introduced the objects which will study in this subsection. Namely, the entangled sets. Readers interested in learning more about them may also look at [20], [43], [74], [118], [121] and [134].

Definition 3.2.28 (Realization)**.** Let $k \in \omega$ and $t : k \longrightarrow \{<, >\}$. Given a total order $(X, <)$ and $a, b \in [X]^k$, we say that (a, b) *realizes* t if

$$a(i)\, t(i)\, b(i)$$

for each $i < k$. By $T(a, b)$ we denote the unique t which is realized by (a, b).

Definition 3.2.29 (Entangled set)**.** Let $(X, <)$ be a partial order, $\mathcal{E} \in [X]^{\omega_1}$ and $k \in \omega$. We say that:

- $\mathcal{E}$ is k-*entangled* if for each uncountable family $\mathcal{A} \subseteq [\mathcal{E}]^k$ of pairwise disjoint sets and $t : k \longrightarrow \{<, >\}$ there are distinct $a, b \in \mathcal{A}$ for which that $T(a, b) = t$.

- $\mathcal{E}$ is *entangled* if it is k-entangled for each $k \in \omega$.

Lemma 3.2.30. *Let $(X, <)$ be a total order, $k \in \omega$ and $\mathcal{E} \in [X]^{\omega_1}$ injectively enumerated as $\langle r_\alpha \rangle_{\alpha \in \omega_1}$. Then $\mathcal{E}$ is k-entangled if and only if for every uncountable family $\mathcal{C} \subseteq [\omega_1]^k$ of pairwise disjoint sets and each $t : k \longrightarrow \{<, >\}$ there are distinct $c, d \in \mathcal{C}$ for which*

$$r_{c(i)} \, t(i) \, r_{d(i)}$$

for each $i < k$.

Proof. As both implications are proved in a completely similar way, we will only show the one from left to right. For this purpose, suppose that $\mathcal{E}$ is k-entangled. Let $\mathcal{C} \subseteq [\omega_1]^k$ be an uncountable family of pairwise disjoint sets and $t : k \longrightarrow \{<, >\}$. Given $c \in \mathcal{C}$, take $h_c : k \longrightarrow k$ the unique function satisfying that $i < j$ if and only if $r_{c(h(i))} < r_{c(h_c(j))}$ for every $i, j < k$. By refining $\mathcal{C}$, we can suppose without loss of generality that there is h for which $h = h_c$ for all $c \in \mathcal{C}$. The key observation is that h codes the increasing enumeration (with respect to order in X) of $\langle r_{c(i)} \rangle_{i<k}$ whenever $c \in \mathcal{C}$. That is, $r_{c(h(0))} < \ldots < r_{c(h(k-1))}$. Now, let $t' = t \circ h$. By the previous observation and since $\mathcal{E}$ is k-entangled, there are distinct $c, d \in \mathcal{C}$ for which

$$T(\langle \, r_{c(h(i))} \, \rangle_{i<k}, \langle \, r_{d(h(i))} \, \rangle_{i<k}) = t'.$$

To finish, take an arbitrary $i < k$ and let $i' = h^{-1}(i)$. Then $r_{c(h(i'))} \, t'(i') \, r_{d(h(i'))}$. But $r_{c(h(i'))} = r_{c(i)}$, $t'(i') = t(i)$ and $r_{d(h(i'))} = r_{d(i)}$. In this way, $r_{c(i)} \, t(i) \, r_{d(i)}$. ∎

Definition 3.2.31. Let $(X, <)$ be a linear order. Given functions $f, g \in X^\omega$, we say that $f <_{lex} g$ if $f(n) < g(n)$ where $n = \min(\, k \in \omega \, : \, f(k) \neq g(k) \,)$.

Remark 3.2.32. $(X^\omega, <_{lex})$ is a linear order which can be embedded in $\mathbb{R}$ whenever X is countable.

Theorem 3.2.33 (Under FCA)**.** *There is an entangled set.*

Proof. Let $\mathcal{F}$ be a fully capturing construction scheme of type $\langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$ satisfying that $n_{k+1} \geqslant 2^{m_k} + 1$ for each $k \in \omega$.
Given $k \in \omega$, let us enumerate $\mathscr{P}(m_k \backslash r_{k+1})$, possibly with repetitions, as $\langle C_i^k \rangle_{0 < i < n_{k+1}}$
Now, for each $\alpha \in \omega_1$ let $f_\alpha : \omega \longrightarrow \mathbb{Z}$ be given as:

$$f_\alpha(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } \Xi_\alpha(k) = -1 \\ \Xi_\alpha(k) & \text{if } k > 0, \, \Xi_\alpha(k) \geqslant 0 \text{ and } \|\alpha\|_{k-1} \in C_{\Xi_\alpha(k)}^{k-1} \\ -\Xi_\alpha(k) & \text{if } k > 0, \, \Xi_\alpha(k) \geqslant 0 \text{ and } \|\alpha\|_{k-1} \notin C_{\Xi_\alpha(k)}^{k-1} \end{cases}$$

Let $\mathcal{E} = \langle f_\alpha \rangle_{\alpha \in \omega_1}$. We claim that $\mathcal{E}$ is an entangled set in $(\mathbb{Z}^\omega, <_{lex})$. Indeed, let $k \in \omega$, $t : k \longrightarrow \{>, <\}$ and $\mathcal{C} \subseteq [\omega_1]^k$ be an uncountable family of pairwise disjoint sets. As $\mathcal{F}$ is fully capturing, there is $\{c_1, \ldots, c_{n_l-1}\} \in [\mathcal{C}]^{n_l}$ which is captured at some level $l > 0$. We claim that there is $i < n_l$ so that $T(c_0, c_i) = t$. First of all, note that $\Delta(c_0(j), c_s(j))$ for each $s < n_l$ and $j < k$. This means that $f_{c_0(j)}|_l = f_{c_s(j)}|_l$. Furthermore, since the elements of $\mathcal{C}$ are pairwise disjoint, then $\Xi_l(c_s(j)) = \Xi_l(c_s) = s$.

This is due to Corollary 2.4.6. From the two previous facts, it is easy to see that, for any $0 < s < n_l$ and $j \in \omega$, the order between $f_{c_0(j)}$ and $f_{c_s(j)}$ is decided by the value of both functions at l. Now, let us consider $0 < i < n_l$ so that

$$C_i^{l-1} = \{ \, \|c_0(j)\|_{l-1} \, : \, j < k \ and \ t(j) = < \, \}.$$

According to the second and third cases of the definitions of the functions, we have that $0 < i = f_{c_i(j)}(l)$ whenever $t(j) = <$ and $0 > -1 = f_{c_i(j)}(l)$ whenever $t(j) = >$. This implies $T(c_0, c_i) = t$, so we are done. ∎

3.3   Colorings at the uncountable

Ramsey theory was implicitly initiated by Frank P. Ramsey in [87]. In there, he proved a lemma which is now known as "Ramsey's theorem".

Theorem 3.3.1. *For all $2 \leqslant n, k \in \omega$ an each $c : [\omega]^n \longrightarrow k$, there is $A \in [\omega]^\omega$ so that $c|_{[A]^n}$ is constant.*

Informally speaking, Ramsey theory studies relation between order and chaos in the realm of functions. In this context, it is usual to interpret the image of a function c as a set of colors, and the function c itself as a *coloring* of the domain. Classical Ramsey theory is concerned in extending and generalizing the pigeonhole principle into different contexts. This is sometimes interpreted as finding order in chaos. For example, if $c : [\omega]^n \longrightarrow k$ is an arbitrary coloring, the pigeonhole principle would imply that there is an infinite set $X \subseteq [\omega]^n$ so that $c|_X$ is constant. Ramsey's theorem improves this by giving a reduced class of sets which testify the previous conclusion. Namely, the family $\{[A]^n \, : \, A \in [\omega]^\omega\}$.

Classical Ramsey-type theorems over countable domains tend to have finite counterparts. However, this is not the case for the uncountable. For example, fix an injection $f : \omega_1 \longrightarrow \mathbb{R}$ and let $c : [\omega]^2 \longrightarrow 2$ be given by:

$$c(a) = \begin{cases} 0 & if \ f(a(0)) < f((a(1)) \\ 1 & otherwise \end{cases}$$

Then there is no $A \in [\omega_1]^{\omega_1}$ so that $c|_{[A]^2}$ is constant. Thus, the natural generalization of Ramsey's theorem to ω_1 fails even when $n = k = 2$. However, in this case there still exists a useful weakening of theorem. Before stating it, we recall some notation.

Definition 3.3.2. Let κ be a (possibly finite) cardinal, X, Y be two arbitrary sets and $c : [X]^\kappa \longrightarrow Y$. Given $A \subseteq X$ and $y \in Y$, we say that A is y-monochromatic if $c|_{[A]^\kappa}$ is constant with value y.

Definition 3.3.3. Let α, β and γ be ordinals. The partition relation $\gamma \to (\alpha, \beta)_2^2$ stands for the following statement:

For all $c : [\gamma]^2 \longrightarrow 2$, there is a 0-monochromatic subset of γ of order type α or there is a 1-monochromatic subset of γ of order type β.

Its negation is written as $\gamma \nrightarrow (\alpha, \beta)_2^2$.

The following theorem is due to Paul Erdös and Richard Rado (see [35]). It is a generalization of a well known theorem of Ben Dushnik, Ernest Miller and Erdös (see [32]). The reader can find a proof in [54] and [64].

Theorem 3.3.4. $\omega_1 \to (\omega_1, \omega + 1)_2^2$.

In [116], Stevo Todorčević proved that it is consistent to have $\omega_1 \to (\omega_1, \alpha)_2^2$ for each $\alpha < \omega_1$. On the other hand, he showed in [121] that $\mathfrak{b} = \omega_1$ implies that $\omega_1 \nrightarrow (\omega_1, \omega + 2)_2^2$. In the following theorem, we will prove that the same is true under CA_2. Although a coloring testifying the negation of the previous partition relation can be extracted from some of our other constructions, we decided to include a direct proof due to its simplicity.

Theorem 3.3.5 (Under CA_2). $\omega_1 \nrightarrow (\omega_1, \omega + 2)_2^2$.

Proof. Let $\mathcal{F}$ be a 2-capturing construction scheme of an arbitrary type. Let $c : [\omega_1]^2 \longrightarrow 2$ defined as:

$$c(\alpha, \beta) = \begin{cases} 1 & \text{if } \Delta(\alpha, \beta) = \rho(\alpha, \beta) \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{F}$ is 2-capturing, it is easy to see that there are no uncountable 0-monochromatic subsets of ω_1. Suppose towards a contradiction that there is a 1-monochromatic set, say X, of order type $\omega + 2$. Let β and γ be the last two elements of X and consider $\alpha \in X \backslash (\gamma)_{\rho(\beta,\gamma)}$. Since ρ is an ordinal metric and $\alpha < \beta < \gamma$ then $\rho(\alpha, \beta) = \rho(\alpha, \gamma)$. In this way, $\Delta(\beta, \gamma) = \rho(\beta, \gamma) < \rho(\alpha, \beta) = \Delta(\alpha, \beta)$. By Lemma 2.3.4, we conclude that $\Delta(\alpha, \gamma) = \Delta(\beta, \gamma)$. Thus, $\rho(\alpha, \gamma) > \Delta(\alpha, \gamma)$ which means that $c(\alpha, \gamma) = 0$. This is a contradiction to X being 1-monochromatic, so we are done. ∎

In contrast to classical Rasmey theory, polychromatic Ramsey theory is about finding chaos. An example of this can be found in [120], where Stevo Todorčević showed the complete failure of Ramsey's theorem at ω_1 when we consider an infinite or uncountable amount of colors. Specifically, he proved the following theorem.

Theorem 3.3.6. *There is a coloring* $c : [\omega_1]^2 \longrightarrow \omega_1$ *such that* $c[[A]^2] = \omega_1$ *for each* $A \in [\omega_1]^{\omega_1}$.

For this, he first constructed a function $\phi : [\omega_1]^2 \longrightarrow \omega_1$ with the property that $\phi[[A]^2]$ contains a closed an unbounded subset (club) of ω_1 for each uncountable A. This function was constructed through the use of an ordinal metric. In the following proposition we show how to define such function from a construction scheme. It is worth pointing out that the same proposition holds for arbitrary ordinal metrics. However the proof is slightly different.

Proposition 3.3.7. *Let* $\mathcal{F}$ *be a construction scheme and* $\phi : [\omega_1]^2 \longrightarrow \omega_1$ *be given as:*

$$\phi(\alpha, \beta) = \min((\beta)_{\Delta(\alpha,\beta)} \backslash \alpha).$$

Then $\phi[[A]^2]$ *contains a club for each* $A \in [\omega_1]^{\omega_1}$.

Proof. Let A be as in the hypotheses. In order to show that $\phi[[A]^2]$ contains a club, it is enough to show that if M is a countable elementary submodel of some $H(\lambda)$ and $A \in M$, then $\delta_M = M \cap \omega_1 \in \phi[[A]^2]$. For each $\alpha \in A \backslash \delta_M$ let $k_\alpha = \rho(\delta_M, \alpha)$. Since A is uncountable there is $B \in [A]^{\omega_1}$ and $k \in \omega$ so that $k_\alpha = k$ for each $\alpha \in B$. By refining B even more, we may assume without loss of generality that $\|\alpha\|_k = \|\beta\|_k$ for any two given $\alpha, \beta \in B$. Let us fix $\beta < \gamma \in B$ and consider $l = \Delta(\beta, \gamma)$. Note that $l > k$ because $\|\beta\|_k = \|\gamma\|_k$. By elementarity, there is $\alpha \in M$ for which $\alpha > \max((\beta)_l \cap \delta_M)$ and $\|\alpha\|_l = \|\gamma\|_l$. Particularly, this means that $\Delta(\alpha, \beta) = l$. It is straightforward that $\delta_M = \min((\beta)_l \backslash \alpha)$. Thus, the proof is over. ∎

In order to prove Theorem 3.3.6, the trick is to take a function $h : \omega_1 \longrightarrow \omega_1$ so that $h^{-1}[\{\xi\}]$ is stationary for any $\xi \in \omega_1$. The function $c = h \circ \phi$ satisfies the conclusion of the theorem.

3.3.1 ccc-polychromatic coloring

In this subsection we build another coloring of the pairs of ω_1 using the capturing axiom CA_3. Before discussing the context of this construction, we give some necesary definitions.

Definition 3.3.8. Let $c : [\omega_1]^2 \longrightarrow \omega_1$ be a coloring, $A \subseteq \omega_1$ and κ be a (possibly finite) cardinal. We say that:

- c is κ-bounded if $|c^{-1}[\{\xi\}]| < \kappa$ for each $\xi \in \omega_1$.

- A is injective if $c|_{[A]^2}$ is injective.

The problem whether every 2-bounded coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ has an uncountable injective sets was first asked by F. Galvin in the early 1980's, who proved that CH implies a negative answer to that question. In [116], Stevo Todorčevic showed that it is consistent, and in particular that PFA implies that every ω-bounded $c : [\omega_1]^2 \longrightarrow \omega_1$ has an uncountable injective set.
In [2], U. Abraham, J. Cummings and C. Smyth proved that it is consistent that there is a 2-bounded coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ without uncountable injective sets in any *ccc* forcing extension. After hearing this theorem, S. Friedman asked for a concrete example of a 2-bounded coloring without an uncountable injective set, but which adquires one in a *ccc* forcing extension (*ccc*-destructible). Such example was produced in [2] assuming CH and the existence of a Suslin tree. Here, we construct one using 3-capturing construction schemes.

Theorem 3.3.9 (Under CA_3). *There is a coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ with the following properties:*

(1) *c is 2-bounded,*

(2) *c has no uncountable injective sets,*

(3) *c is ccc-destructible.*

Proof. Let $\mathcal{F}$ be a 3-capturing construction scheme of an arbitrary type. Let $\psi : \omega_1 \times \omega \times \omega \longrightarrow \omega_1$ be a bijection. We define $c : [\omega_1]^2 \longrightarrow \omega$ as follows:

$$c(\alpha, \beta) = \begin{cases} \psi(\beta, \rho(\alpha, \beta), \|\alpha\|_{\rho(\alpha,\beta)}) & \text{if } \alpha < \beta \text{ and } \Xi_\beta(\rho(\alpha, \beta)) > 2 \\ \psi(\beta, \rho(\alpha, \beta), \|\alpha\|_{\rho(\alpha,\beta)-1}) & \text{if } \alpha < \beta \text{ and } \Xi_\beta(\rho(\alpha, \beta)) \leqslant 2 \end{cases}$$

In the following three claims we prove that c satisfies the conclusions of the theorem.

Claim 1: c is 2-bounded.

Proof. Let $\xi \in \omega_1$ and suppose that $\{\alpha_0, \beta_0\}, \{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ are elements of $c^{-1}[\{\xi\}]$. We will show that two of these pairs are equal. For this purpose, take $\beta \in \omega_1$ and $k, a \in \omega$ for which $\phi^{-1}(\xi) = (\beta, k, a)$. Given $i < 3$ we have that $\psi^{-1} \circ c(\alpha_i, \beta_i) = (\beta, k, a)$. This means that $\beta_i = \beta$ and $\rho(\alpha_i, \beta_i) = k$. Since ρ is an ordinal metric,

$$\rho(\alpha_i, \alpha_j) \leqslant \max(\rho(\alpha_i, \beta), \rho(\alpha_j, \beta)) = k$$

for any $i, j < 3$. In order to finish the proof of the claim we have to consider two cases. The first one is when $\Xi_\beta(k) > 2$. Here, $\|\alpha_i\|_k = a$ for each i. Particularly, $\Delta(\alpha_0, \alpha_1) > k$. But $\rho(\alpha_0, \alpha_1) \leqslant k$. The only way in which this is possible is if $\alpha_0 = \alpha_1$. So this case is over. The remaining case is when $\Xi_\beta(k) < 2$. According to the part (b) of Lemma 2.3.9, $0 \leqslant \Xi_{\alpha_i}(k) < \Xi_\beta(k) \leqslant 2$ for each i. Therefore, there are $i < j < 3$ for which $\Xi_{\alpha_i}(k) = \Xi_{\alpha_j}(k)$. Hence, by the point (d) of Lemma 2.3.9, $\Delta(\alpha, \beta) \neq k$. As $\|\alpha_i\|_{k-1} = a = \|\alpha_j\|_{k-1}$, we conclude that $\Delta(\alpha_i, \alpha_j) > k \geqslant \rho(\alpha_i, \alpha_j)$. In this way, $\alpha_i = \alpha_j$. $\square$

Claim 2: c has no uncountable injective set.

Proof. Now, we prove c has no uncountable injective set. For this, let $S \in [\omega_1]^{\omega_1}$. Since $\mathcal{F}$ is 3-capturing, there is $\{\alpha_0, \alpha_1, \alpha_2\} \in [S]^3$ which is captured at some level $l \in \omega$. In particular, $\Xi_{\alpha_2}(l) = 2$ and for each $i < 2$ the following properties hold:

- $\rho(\alpha_i, \alpha_2) = l$,

- $\|\alpha_i\|_{l-1} = \|\alpha_2\|_{l-1}$.

In other words, $c(\alpha_0, \alpha_2) = c(\alpha_1, \alpha_2)$. $\square$

Claim 3: c is *ccc*-destructible.

Proof of claim. Let $\mathbb{P} = \{p \in [\omega_1]^{<\omega} : p \text{ is injective}\}$ ordered by reverse inclusion. We claim that $\mathbb{P}$ is *ccc*. Since $\mathcal{F}$ is 3-capturing (thus, 2-capturing), if $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$ there are $p, q \in \mathcal{A}$, $l \in \omega$ and $F \in \mathcal{F}_l$ capturing p and q. By definition of c, it is easy to see that $p \cup q$ is injective. Hence, $\mathcal{A}$ is not an antichain, and since $\mathcal{A}$ was arbitrary, $\mathbb{P}$ is *ccc*. Finally, since $\mathbb{P}$ is *ccc* and uncountable, there is $p \in \mathbb{P}$ which forces the generic filter to be uncountable. From this it follows that if G is a $\mathbb{P}$-generic filter over V containing p, then $\bigcup G$ is an uncountable injective set. Thus, the proof is over. $\square$

■

3.4 $\quad$ Oscillation theory of 2-capturing schemes

In [121], Stevo Todorčević developed powerful Ramsey-type results by analyzing the behaviour of unbounded families of functions of ω^ω. From these results, he deduced very interesting theorems in a great variety of areas. His analysis was based in the following concept.

Definition 3.4.1 (Oscillation of functions)**.** Let $f, g \in \omega^\omega$ and $k \in \omega$. We define the *k-oscillation set of f to g* as

$$\overline{osc}_k(f, g) = \{s \in \omega \setminus k \, : \, f(s) \leqslant g(s) \text{ and } f(s+1) > g(s+1)\},$$

Additionally, we define the *oscillation number of f to g* as $osc_k(f, g) = |\overline{osc}_k(f, g)|$.

Remark 3.4.2. Suppose that $f, g \in \omega^\omega$ are so that $f \neq^* g$. If f and g are not comparable with respect to $<^*$ then $osc_k(f, g) = \omega = osc_k(g, f)$. However, if f and g are comparable, it is not necessarily true in general that $osc(f, g) = osc(g, f)$.

$\quad$ In this section we will prove that 2-capturing construction schemes can be used to define bounded families of functions whose properties reassemble the ones from unbounded families. In particular, we will be able to show that CA_2 implies a lot of things that are also implied by the hypothesis $\mathfrak{b} = \omega_1$. This is an interesting phenomenom as CA_2 is independent from the previous assumption. The reader interested in knowing about other oscillation theories may look for [108].

$\quad$ For the rest of this section we fix $\mathcal{F}$ a 2-capturing construction scheme of some type $\langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$. We will analyse the behaviour of the oscillation number associated to the k-cardinality functions associated to $\mathcal{F}$. Given $\alpha \in \omega_1$ let $f_\alpha : \omega \longrightarrow \omega$ be given as:

$$f_\alpha(l) = \|\alpha\|_l.$$

We define $\mathcal{B}_\mathcal{F}$ as $\langle f_\alpha \rangle_{\alpha \in \omega_1}$. The following lemma is a direct consequence of the definitions of ρ, Δ and Lemma 2.3.9.

Lemma 3.4.3. *Let $\alpha < \beta \in \omega_1$. Then:*

(1) $f_\alpha(i) = f_\beta(i)$ *if $i < \Delta(\alpha, \beta)$,*

(2) $f_\alpha(j) < f_\beta(j)$ *whenever $j \geqslant \rho(\alpha, \beta)$,*

(3) $f_\alpha < f_\beta$ *provided that $\Delta(\alpha, \beta) = \rho(\alpha, \beta)$*

(4) *In particular, $f_\alpha <^* f_\beta$.*

Furthermore, $\mathcal{B}_\mathcal{F}$ is bounded by the function in ω^ω which sends each i to m_i.

$\quad$ It is interesting that even though $\mathcal{B}_\mathcal{F}$ is bounded, its oscillation theory mirrors the oscillation theory of [121] for unbounded families. Since $\mathcal{F}$ is 2-capturing, given any $\mathcal{A} \in [\mathcal{B}_\mathcal{F}]^{<\omega}$, there are $\alpha < \beta \in \mathcal{A}$ with $\Delta(\alpha, \beta) = \rho(\alpha, \beta)$. Thus, we have the following corollary.

Corollary 3.4.4. $(\mathcal{B}_{\mathcal{F}}, \leqslant)$ *has no uncountable pies*[5].

Given $\alpha, \beta \in \omega_1$ and $k \in \omega$ we will write $osc_k(\alpha, \beta)$ and $\overline{osc}_k(\alpha, \beta)$ instead of $osc_k(f_\alpha, f_\beta)$ and $\overline{osc}_k(f_\alpha, f_\beta)$ respectively. These two objects will be written as $osc(\alpha, \beta)$ and $\overline{osc}(\alpha, \beta)$ whenever $k = 0$.

Definition 3.4.5. Let $a, b \in \mathrm{FIN}(\omega_1)$ and $k \in \omega$. We define the oscillation from a to b as
$$osc_k[a, b] = \{osc_k(\alpha, \beta) \mid \alpha \in a \ and \ \beta \in b\}.$$

Proposition 3.4.6. *Let $n, k \in \omega$ and $\mathcal{A} \in [\omega_1]^n$ be an uncountable family of pairwise disjoint sets such that $\rho^a = k$ for each $a \in \mathcal{A}$. Given $l \in \omega$, there are $a < b \in \mathcal{A}$ such that $osc_k[a, b] \subseteq [l, 2l]$.*

Proof. The proof is by induction over l.

Base step: Suppose that $l = 0$. Since $\mathcal{F}$ is 2-capturing, there are $a < b \in \mathcal{A}$ so that the pair $\{a, b\}$ is captured at some level $s > k$. In particular, for each $\rho(a(i), b(i)) = s = \Delta(a(i), b(i))$ and $\rho(a(i), b(j)) = s$ for all distinct $i, j < k$. According to by Lemma 3.4.3 and the previous observation, the following properties hold for all $i, j < k$:

(1) $f_{a(i)}|_{[k,s)} = f_{b(i)}|_{[k,s)}$,

(2) $f_{a(i)}|_{\omega \setminus s} < f_{b(j)}|_{\omega \setminus s}$,

(3) $f_{a(i)}|_{[k,\rho^F)} < f_{a(j)}|_{[k,\rho^F)}$ provided that $i < j$.

From the previous facts, we conclude that $osc[a, b] = \{0\} = [l, 2l]$.

Inductive step: Suppose that we have proved the proposition for some $l \in \omega$ and let $\mathcal{A}$ be as in the hypotheses. Using the inductive hypotheses, we can recursively construct an uncountable $\mathcal{C} \subseteq [\mathcal{A}]^2$ of pairwise disjoint sets so that for each $\{a, b\} \in \mathcal{C}$ the following conditions hold:

- Either $a < b$ or $b < a$.[6]

- if $a < b$ then $osc_k[a, b] \subseteq [l, 2l]$.

From this point on, whenever say that $\{a, b\} \in \mathcal{C}$ we will assume that $a < b$. Since $\mathcal{F}$ is 2-capturing, we can find an uncountable family $\mathcal{D} \subseteq [\mathcal{C}]^2$ and $r > k + 1$ with the following properties:

- Whenever $\{\{a, b\}, \{c, d\}\} \in \mathcal{D}$, the pair $\{a \cup b, c \cup d\}$ is captured at level r. In particular, this implies that $r > \rho^{a \cup b}$ and $a \cup b < c \cup d$.

- For each $x, y \in D$, $\bigcup x \cap \bigcup y = \emptyset$.

---

[5]Recall that by pie, we mean a set of pairwise incomparable elements
[6]Recall that $a < b$ means that $\max(a) < \min(b)$.

Using once again that $\mathcal{F}$ is 2-capturing, we can get $s > r$ and two elements of $\mathcal{D}$, say $\{\{a_0, b_0\}, \{c_0, d_0\}\}, \{\{a_1, b_1\}, \{c_1, d_1\}\}$, for which the pair

$$\{(a_0 \cup b_0) \cup (c_0 \cup d_0), (a_1 \cup b_1) \cup (c_1 \cup d_1)\}$$

is captured at level s. We will finish by proving the following claim.

<u>Claim:</u> $osc_k[c_0, b_1] \subseteq [l + 1, 2(l + 1)]$

Proof of claim. For this, take $i, j < n$. The following properties follow from Lemma 3.4.3:

(1) $f_{b_1(j)}|_{[k,s)} = f_{b_0(j)}|_{[k,s)}$. This is because, in particular, $\{b_0, b_1\}$ is captured at level s.

(2) $f_{c_0(i)}|_{[k,r)} = f_{a_0(i)}|_{[k,r)}$. This is due to the fact that $\{\{a, b\}, \{c, d\}\} \in \mathcal{D}$. That is, the pair $\{a_0 \cup b_0, c_0 \cup d_0\}$ is captured at level r.

(3) $f_{b_1(j)}|_{\omega \setminus s} > f_{c_0(i)}|_{\omega \setminus s}$. This is true since $\rho(b_1(j), c_0(i)) = s$ and $c_0 < b_1$.

(4) $f_{b_0(j)}|_{[r,\rho^F)} < f_{c_0(i)}|_{[r,\rho^F)}$. Similarly to the previous point. This inequality holds because $\rho(b_0(j), c_0(i)) = r$ and and $b_0 < c_0$.

We will use these properties to calculate the oscillation. First observe that we can use the part (2) of Lemma 3.4.3 to conclude $\overline{osc}_k(a_0(i), b_0(j)) \subseteq [k, \rho^{a_0 \cup b_0})$ and

$$\overline{osc}_k(c_0(i), b_1(j)) \subseteq [k, \rho^F).$$

According the properties (1) and (2) written above and since $r > \rho^{a_0 \cup b_0}$,

$$\overline{osc}_k(c_0(i), b_1(j)) \cap [k, r - 1) = \overline{osc}_k(a_0(i), b_0(j)).$$

Due to properties (1), (3) and (4) we also have that $\rho^F - 1 \in \overline{osc}_k(c_0(i), b_1(j))$. In fact, properties (1) and (4) also imply that $\rho^F - 1$ is the only element in the interval $[r, \rho^F)$ which belong to $\overline{osc}_k(c_0(i), b_1(j))$. By joining all the previous observations, we get that:

$$\overline{osc}_k(a_0(i), b_0(j)) \cup \{\rho^F - 1\} \subseteq \overline{osc}_k(c_0(i), b_1(j)) \subseteq \overline{osc}_k(a_0(i), b_0(j)) \cup \{\rho^F - 1\} \cup \{r - 1\}.$$

This means that $l + 1 \leqslant osc_k(c_0(i), b_1(j)) \leqslant 2l + 2$. Thus, the proof is complete. $\square$

$\blacksquare$

By a careful analysis of the argument of the preceding theorem, one can show that whenever $\mathcal{A} \in [\omega_1]^\omega$ and $l \in \omega$, then there are $\alpha < \beta \in \mathcal{A}$ for which $osc(\alpha, \beta) = l$. Unfortunately, this property does not hold for arbitrary uncountable families of finite sets. Nevertheless, the previous result is enough to redefine a "corrected" oscillation.

The following lemma is easy.

Lemma 3.4.7. *There is a partition $\langle P_n \rangle_{n \in \omega}$ of ω such that for every $k, n \in \omega$ there is $l \in \omega$ such that $[l, 2l + k] \subseteq P_n$.*

Theorem 3.4.8 (Under CA_2). *There is a coloring $o : [\omega_1]^2 \longrightarrow \omega$ such that for every uncountable family $\mathcal{A} \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets and each $n \in \omega$, there are $a < b \in \mathcal{A}$ for which $\{o(\alpha, \beta) : \alpha \in a \text{ and } \beta \in b\} = \{n\}$.*

Proof. Let $\langle P_n \rangle_{n \in \omega}$ be a partition of ω as in Lemma 3.4.7. Let $o : [\omega_1]^2 \longrightarrow \omega$ be defined as:

$$o(\alpha, \beta) = n \text{ if and only if } osc(\alpha, \beta) \in P_n.$$

We claim that o satisfies the conclusion of the theorem. Indeed, let $\mathcal{A}$ be an uncountable family of pairwise disjoint finite subsets of ω_1 and $n \in \omega$. By refining $\mathcal{A}$ we may suppose that there is $k \in \omega$ such that $\rho^a = k$ for every $a \in \mathcal{A}$. Let $l \in \omega$ be such that $[l, 2l + k] \subseteq P_n$. Due to Proposition 3.4.6, there are $a < b \in \mathcal{A}$ such that $osc_k(a, b) \subseteq [l, 2l]$. Given $\alpha \in a$ and $\beta \in b$, it is easy to see that $\overline{osc}(\alpha, \beta)$ has at most k more elements than $\overline{osc}_k(\alpha, \beta)$. In this way, $osc(\alpha, \beta) \in [l, 2l + k] \subseteq P_n$. In other words, $o(\alpha, \beta) = n$. This finishes the proof. ∎

The existence of a coloring with the properties stated above, already implies the existence of a much more powerful coloring. As we shall mention later, such a coloring can be used to build topological spaces with important properties.

Corollary 3.4.9 (CA_2). *There is a coloring $o^* : [\omega_1]^2 \longrightarrow \omega$ such that for all $n \in \omega$, $h : n \times n \longrightarrow \omega$ and any uncountable family $\mathcal{A} \subseteq [\omega_1]^n$ of pairwise disjoint sets, there are $a < b \in \mathcal{A}$ for which*

$$o^*(a(i), b(j)) = h(i, j)$$

for all $i, j < n$.

Proof. Let $\langle h_n \rangle_{n \in \omega}$ be an enumeration of all $h : X \longrightarrow \omega$ for which $X \subseteq \omega^{<\omega}$ is finite and its elements are pairwise incomparable. Let us call X_n the domain of h_n. Note that for each $f \in \omega^\omega$ and every $n \in \omega$ there is at most one $\sigma \in X_n$ which is extended by f. Take a coloring o satisfying the conclusion of Theorem 3.4.8. We define $o^* : [\omega_1]^2 \longrightarrow \omega$ as follows: Given distinct $\alpha, \beta \in \omega_1$, if there are $\sigma_\alpha, \sigma_\beta \in X_{o(\alpha,\beta)}$ for which $\sigma_\alpha \subseteq f_\alpha$ and $\sigma_\beta \subseteq f_\beta$, put

$$o^*(\alpha, \beta) = h_{o(\alpha,\beta)}(\sigma_\alpha, \sigma_\beta).$$

In any other case, let $o^*(\alpha, \beta) = 17$. In order to prove that o^* satisfies the conclusion of the corollary, let $n \in \omega$, $h : n \times n \longrightarrow \omega$ and $\mathcal{A} \subseteq [\omega_1]^n$ be an uncountable family of pairwise disjoint sets. By refining $\mathcal{A}$ we may suppose there is $k \in \omega$ with the following properties:

(1) $\forall a \in \mathcal{A} \, \forall i \neq j < n \, (f_{a(i)}|_k \neq f_{a(j)}|_k)$,

(2) $\forall a, b \in \mathcal{A} \, \forall i < n \, (f_{a(i)}|_k = f_{b(i)}|_k)$.

Fix $a_0 \in \mathcal{A}$. Let $X = \{f_{a_0(i)}|_k : i < n\}$ and define $h : X \times X \longrightarrow \omega$ as:

$$h(f_{a_0(i)}|_k, f_{a_0(j)}|_k) = h(i, j).$$

We know that there is $m \in \omega$ for which $X = X_m$ and $h = h_m$. For such m, there are $a < b \in \mathcal{A}$ for which $o(a(i), b(j)) = m$ for all $i, j < n$. For all such i and j, $f_{a_0(i)}|_k \subseteq f_{a(i)}$

and $f_{a_0(j)}|_k \subseteq f_{a(j)}$. In this way, $o^*(a(i), b(j)) = h(f_{a_0(i)}|_k, f_{a_0(j)}|_k) = h(i, j)$. So we are done.

∎

As an application, we get the following:

Corollary 3.4.10 (CA_2)**.** *ccc is not productive.*

Proof. Let o be a coloring of Theorem 3.4.8. For each $n \in \omega$, let

$$\mathbb{P}_n = \{p \in [\omega_1]^{<\omega_1} : \forall \alpha, \beta \in p \, (\text{if } \alpha \neq \beta \text{ then } o(\alpha, \beta) = n)\}.$$

In particular, $\mathbb{P}_0$ and $\mathbb{P}_1$ are *ccc* but $\mathbb{P}_0 \times \mathbb{P}_1$ is not. The set $\{(\alpha, \alpha) \,|\, \alpha \in \omega_1\}$ testifies this last fact. ∎

3.4.1 A sixth Tukey type

The purpose of this subsection is to show that the family $\mathcal{B}_{\mathcal{F}}$ is a so called sixth Tukey type. Before we begin, recall that a partial order $(D, \leqslant)$ is *(upwards) directed* if for every $x, y \in X$ there is $z \in D$ bigger than x and y.

Proposition 3.4.11. $(\mathcal{B}_{\mathcal{F}}, \leqslant)$ *is directed.*

Proof. Let $\alpha < \beta \in \omega_1$ and let $F \in \mathcal{F}_{\rho(\alpha, \beta)}$ be such that $\{\alpha, \beta\} \subseteq F$. Observe that if $\delta = \max F$ then $f_\delta(i) = m_i$ for each $i \leqslant \rho(\alpha, \beta)$. Moreover, $\rho(\delta, \alpha), \rho(\delta, \beta) \leqslant \rho(\alpha, \beta)$ as testified by F. Hence, for each $i > \rho(\alpha, \beta)$, $f_\alpha(i), f_\beta(i) \leqslant f_\delta(i)$ by means of the point (2) of the Lemma 3.4.3. From the previous two observations we conclude that $f_\alpha, f_\beta < f_\delta$. ∎

The following concept was introduced by John W. Tukey in [124]. Among other things, he proved that the sets 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$ are non Tukey equivalent when equipped with their natural orderings.

Definition 3.4.12 (The Tukey ordering)**.** Let $(D, \leqslant_D)$ and $(E, \leqslant_E)$ be directed partial orders. We say that E is *Tukey below* D, and write it as $E \leqslant_T D$ if there is $\phi : D \longrightarrow E$ such that $\phi[X]$ is cofinal in E for each cofinal $X \subseteq D$. Furthermore, we say that E is *Tukey equivalent* to D, and write it as $E \equiv_T D$, if $E \leqslant_T D$ and $D \leqslant_T E$.

Remark 3.4.13. In view of Tukey's result We will refer to the orders 1, ω, ω_1, $\omega_1 \times \omega_1$ and $[\omega_1]^{<\omega}$ as the *five canonical Tukey types*.

In [40], John R. Isbell showed that under CH, there is at least one directed partial order of cardinality ω_1 which is non Tukey equivalent to none of the five canonical Tukey types. He later improved his result in [53]. In [121], Stevo Todorčević proved the existence of such a directed partial order under the hypothesis $\mathfrak{b} = \omega_1$. In [117], he proved that consistently every directed partial order of cardinality ω_1 is Tukey equivalent to one of the five canonical Tukey types.

Theorem 3.4.14 (Under PFA)**.** *Let $(D. \leqslant)$ be a directed set of cardinality ω_1. Then D is Tukey equivalent to 1, ω, ω_1, $\omega \times \omega_1$ or $[\omega_1]^{<\omega}$.*

Remark 3.4.15. From now on, we will call such an order, a *sixth Tukey type*.

The reader interested in learning more about the Tukey ordering and related topics is invited to search for [29], [30], [45], [48], [70], [79], [85], [98] and [99].

The following Proposition can be found in [124].

Proposition 3.4.16. *Let $(D, \leqslant_D)$ and $(E, \leqslant_E)$ be directed partial orders.*

- $E \leqslant_T D$ *if and only if there is $\phi : E \longrightarrow D$ such that $\phi[X]$ is unbounded in D for each unbounded $X \subseteq E$.*

- $E \equiv_T D$ *if and only if there is a partially ordered set C in which both D and E can be embedded as cofinal subsets.*

Definition 3.4.17. Let $(D, \leqslant)$ be a directed partial order. We say $S \subseteq D$ is ω-bounded if every countable subset of S is bounded in D.

The following proposition appeard in [117].

Proposition 3.4.18. *Let $(D, \leqslant)$ be a directed set with $|D| = \omega_1$. Then:*

(1) *$D \leqslant_T 1$ if and only if D has a greatest element.*

(2) *$D \leqslant_T \omega$ if and only if $cof(D) \leqslant \omega$.*

(3) *$D \leqslant \omega_1$ if and only if D is ω-bounded.*

(4) *$D \leqslant_T \omega \times \omega_1$ if and only if D can be covered by countably many ω-bounded sets.*

(5) *$[\omega_1]^{<\omega} \leqslant_T D$ if and only if there is $A \in [D]^{\omega_1}$ for which every $X \in [A]^\omega$ is unbounded in D.*

As an easy consequence of the previous result, we have the following.

Corollary 3.4.19. *Let $(D, \leqslant)$ be a directed set with $|D| = \omega_1$. Then either $D \equiv_T 1$, $D \equiv_T \omega$, $D \equiv_T \omega_1$ or $\omega \times \omega_1 \leqslant_T D \leqslant_T [\omega_1]^\omega$.*

Now, we are ready to prove that $\mathcal{B}_{\mathcal{F}}$ is a sixth Tukey type. It is worth mentioning that this proof is completely similar to the one under $\mathfrak{b} = \omega_1$.

Theorem 3.4.20. *$(\mathcal{B}_{\mathcal{F}}, \leqslant)$ is a sixth Tukey type.*

Proof. By Corollary 3.4.19, it is enough to show $\mathcal{B}_{\mathcal{F}} \not\leqslant_T \omega \times \omega_1$ and $[\omega_1]^{<\omega} \not\leqslant_T \mathcal{B}_{\mathcal{F}}$. This will be a consequence of the next two claims, and due to the points (4) and (5) of Proposition 3.4.18.

<u>Claim</u>: $\mathcal{B}_{\mathcal{F}}$ does not contain any uncountable ω-bounded set.

Proof of claim. For this, we argue by contradiction. Assume there is $A \in [\omega_1]^{\omega_1}$ for which $\langle f_\beta \rangle_{\beta \in A}$ is ω-bounded. Recursively, we can build a sequence $\langle \alpha_\xi, \beta_\xi \rangle_{\xi < \omega_1}$ of pairs of countable ordinals satisfying the following properties:

(1) $\langle \alpha_\xi \rangle_{\xi \in \omega_1}$ and $\langle \beta_\xi \rangle_{\xi \in \omega_1}$ are increasing,

(2) $\beta_\xi < \alpha_\xi$ for any $\xi \in \omega_1$,

(3) $\langle \alpha_\xi \rangle_{\xi \in \omega_1} \subseteq A$,

(4) for each $\xi \in \omega_1$, f_{β_ξ} is an upper bound of the set $\{ f_{\alpha_\nu} : \nu < \xi \}$.

Since $\mathcal{F}$ is 2-capturing, $\delta < \gamma$ so that $\{\{\beta_\delta, \alpha_\delta\}, \{\beta_\gamma, \alpha_\gamma\}\}$ is captured at some level $l \in \omega$. It follows that $f_{\beta_\gamma}(l-1) = f_{\beta_\delta}(l-1) < f_{\alpha_\delta}(l-1)$. But this is a contradiction since f_{β_γ} was supposed to bound f_{α_δ}. $\qquad \square$

<u>Claim</u>: Let $A \in [\omega_1]^{\omega_1}$. Then $\langle f_\alpha \rangle_{\alpha \in A}$ contains an infinite bounded set.

Proof of claim. Let us define the coloring $d : [A]^2 \longrightarrow 2$ as:

$$ d(\alpha, \beta) = \begin{cases} 0 & \text{if } f_\alpha \not\leqslant f_\beta \text{ and } f_\beta \not\leqslant f_\alpha \\ 1 & \text{otherwise} \end{cases} $$

By Theorem 3.3.4, there are two possibilities:

(A)

A contains a 0-monochromatic uncountable set.

(B)

A contains a 1-monochromatic set of order type $\omega + 1$.

Every 0-monochromatic set is an antichain in $\mathcal{B}_\mathcal{F}$, so by Corollary 3.4.4 there can not be uncountable 0-monochromatic sets. Hence, there is 1-monochromatic subset of A, say X, of order type $\omega + 1$. Observe that f_β bounds $\langle f_\alpha \rangle_{\alpha \in X}$ where $\beta = \max(X)$. $\qquad \square$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\blacksquare$

3.4.2 A Suslin tower

Definition 3.4.21. Let $\mathcal{T}$ be an κ-tower. We say $\mathcal{T}$ is *Suslin* if for every uncountable $\mathcal{A} \subseteq \mathcal{T}$ there are distinct $A, B \in \mathcal{A}$ with $A \subseteq B$.

Suslin towers were studied in [13]. There, Piotr Borodulin-Nadzieja and David Chodounský proved, in particular, that Suslin ω_1-towers exist under $\mathfrak{b} = \omega_1$.

Note that whenever $\mathcal{B}$ is an increasing family of functions in ω^ω with respect to $<^*$ of order type ω_1, then the family $\langle T_f \rangle_{f \in \mathcal{B}}$, where

$$ T_f = \{(n, m) : m \leqslant f(n)\} $$

is a Suslin tower. Furthermore, if $f, g \in \mathcal{B}$ are such that $f \leqslant g$, then $T_f \subseteq T_g$. Consequently, if $(\mathcal{B}, \leqslant)$ has no uncountable pies then $\langle T_f \rangle_{f \in \mathcal{B}}$ is Suslin. Thus, we have the following corollary.

Corollary 3.4.22 (CA_2). *There is a Suslin ω_1-tower.*

3.4.3   S and L spaces

In this subsection we will study two of the most famous problems in general topology.

Definition 3.4.23 (S-space and L-space)**.** Let (X, τ) be a topological space. We say that:

- X is an *S-space* if it is T_3, hereditarily separable and not hereditarily Lindelöf. Additionally, we say that X is a *strong S-space* if X^n is an S-space for each $1 \leqslant n \in \omega$.

- X is an *L-space* if it is T_3, hereditarily Lindelöf and not hereditarily separable. Moreover, we say that X is a *strong L-space* if X^n is an L-space for all $1 \leqslant n \in \omega$.

The existence of an S-space and an L-space used to be one of the main open problems in set-theoretic topology. Such spaces exist under a large variety of axioms, like CH and some parametrized diamonds of [78]. The question regarding the existence of an S-space settled when Stevo Todorčevic proved the following theorem.

Theorem 3.4.24 (Under PFA)**.** *There are no S-spaces.*

For some time, people thought that the existence of S-spaces was equivalent to the existence of L-spaces. That is, there is an S-space if and only if there is an L-space. However, this is not the case. By using the technic of walks on ordinals, and some number theory, Justin Moore proved that the existence of an L-space already follows from the usual axioms of Set Theory. This was done in [76].

Theorem 3.4.25. *There is an L-space.*

To learn more about S-spaces (and L-spaces) the reader may consult [1], [89], [90] and [121].

Definition 3.4.26 (Right and Left separated spaces)**.** Let (X, τ) be a topological space with $|X| = \omega_1$ and $\langle x_\alpha \rangle_{\alpha \in \omega_1}$ be an enumeration of X. We say that:

- X is *left-open (right-separated)* if $\{x_\xi : \xi \leqslant \alpha\}$ is open for every $\alpha \in \omega_1$.

- X is *right-open (left-separated)* if $\{x_\xi : \xi \geqslant \alpha\}$ is open for every $\alpha \in \omega_1$.

The following is lemma is well-known. We prove it here for convenience of the reader.

Lemma 3.4.27. *Suppose that (X, τ) is a topological space with the following properties:*

1. *X is T_3,*

2. *X is locally countable,*

3. *X does not have uncountable discrete sets.*

Then X is an S-space

Proof. For each $x \in X$ let U_x be a countable open neighborhood of x. Since X is uncountable it follows that the cover $\{U_x : x \in X\}$ does not have a countable subcover. Hence, X is not Lindelöf. The only thing left to prove is that X is hereditarily separable. For this aim, let $Y \subseteq X$. Suppose towards a contradiction that Y is not separable. It is easy to see that $Y \backslash \overline{D}$ is uncountable for each $D \in [Y]^{\leqslant \omega}$. By using this fact, we can recursively construct a sequence $\langle y_\alpha \rangle_{\alpha \in \omega_1}$ so that

$$y_\beta \notin \overline{\langle y_\alpha \rangle_{\alpha < \beta}} \cup \Big(\bigcup_{\alpha < \beta} U_{y_\alpha} \Big)$$

for each $\beta \in \omega_1$. Note that for any such β, the set $V_\beta = U_{y_\beta} \backslash \overline{\{y_\alpha : \alpha < \beta\}}$ is an open neighborhood of y_β so that $V_\beta \cap \{y_\alpha\}_{\alpha \in \omega} = \{y_\beta\}$. In this way, $\langle y_\alpha \rangle_{\alpha \in \omega_1}$ is an uncountable discrete set. This contradiction ends the proof. ∎

Corollary 3.4.28. *Let (X, τ) be a T_3 topological space and $\langle x_\alpha \rangle_{\alpha \in \omega_1}$ be an enumeration of X. If X is left-open and does not have have uncountable descrete sets then it is is an S-space. Furthermore, if X^n does not have uncountable discrete sets for any $1 \leqslant n \in \omega$, then X is a strong S-space.*

Theorem 3.4.29. *Suppose that there is a coloring $o^* : [\omega_1]^2 \longrightarrow \omega$ satisfying the conclusions of corollary 3.4.9. Then there is a strong S-space.*

Proof. Let o^* be a coloring as in the hypotheses. For each $\alpha \in \omega_1$ define $x_\alpha : \omega_1 \longrightarrow 2$ as follows:

$$x_\alpha(\beta) = \begin{cases} \min(o^*(\alpha, \beta), 1) & \text{if } \alpha < \beta \\ 0 & \text{if } \alpha > \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

Consider $X = \langle x_\alpha \rangle_{\alpha \in \omega_1}$ endowed with the product topology inherited by 2^{ω_1}. It is clear that X is T_3. Furthermore, for each $\alpha \in \omega_1$, the set $\{x_\beta : \beta \in \omega_1 \text{ and } x_\beta(\alpha) = 1\}$ is an open set contained in $\{x_\xi : \xi \leqslant \alpha\}$ and having x_α as an element. Hence, X is also left-open. According to the Corollary 3.4.28, the only thing left to do is to show that none the finite powers of X contain an uncountable discrete set. As we will construct another strong S-space later in this subsection, we will only prove the previous fact for X.

Claim: X has no uncountable discrete set.

Proof of claim. Suppose towards a contradiction that there is $A \in [\omega_1]^{\omega_1}$ so that $\langle x_\alpha \rangle_{\alpha \in \omega_1}$ is discrete. For each $\beta \in A$ let $h_\beta \in \omega_1^{<\omega}$ so that $\beta \in dom(h_\beta)$, $x_\beta \in [h_\beta] = \{x_\alpha : \alpha \in \omega_1 \text{ and } h \subseteq x_\beta\}$ and $[h]_\beta \cap \langle x_\alpha \rangle_{\alpha \in \omega_1} = \{x_\beta\}$. Put $D_\beta = dom(x_\beta)$. By applying usual refining arguments we can assume that for each $\alpha, \beta \in A$ the following conditions hold:

- $D_\alpha < D_\beta$ whenever $\alpha < \beta$,

- $|D_\alpha| = |D_\beta|$. Furthermore, if $\phi : D_\beta \longrightarrow D_\alpha$ is the increasing bijection, then $\phi(\beta) = \alpha$ and $h_\alpha(\phi(\xi)) = h_\beta(\xi)$ for each $\xi \in D_\beta$.

Let m be the common cardinality of each D_β. By the second point above, there is $k \in m$ and $h' : m \longrightarrow 2$ so that $D_\alpha(k) = \alpha$ and $h'(i) = h_\alpha(D_\alpha(i))$ for each $\alpha \in A$. We now define $h : m \times m \longrightarrow 2$ as follows:

$$h(i, j) = \begin{cases} h'(j) & \textit{if } i = k \\ 0 & \textit{otherwise} \end{cases}$$

Since o^* satisfies the conclusions of corollary 3.4.9, there are $\alpha < \beta \in A$ so that

$$o^*(D_\alpha(i), D_\beta(j)) = h(i, j)$$

for each $i, j \in m$. Particularly,

$$h_\beta(D_\beta(j)) = h'(j) = h(k, j) = o^*(D_\alpha(k), D_\beta(j)) = o^*(\alpha, D_\beta(j)) = x_\alpha(D_\beta(j))$$

for any given $j \in m$. Therefore, $x_\alpha \in [h_\beta]$. This is a contradiction, so the claim is over.

$\square$

$\blacksquare$

In the same way, we can define for each $\alpha \in \omega_1$, y_α as follows:

$$y_\alpha(\beta) = \begin{cases} \min(o^*(\alpha, \beta), 1) & \textit{if } \alpha > \beta \\ 0 & \textit{if } \alpha < \beta \\ 1 & \textit{if } \alpha = \beta \end{cases}$$

By a similar argument, one can show that $Y = \langle y_\alpha \rangle_{\alpha \in \omega_1}$ is a strong L-space.

Now, we will build another S-space using the family $\mathcal{B}_\mathcal{F}$.

Definition 3.4.30 (The topology τ_S). Given $\alpha \in \omega_1$, let $C(\alpha) = \{f_\xi : f_\xi \leqslant f_\alpha\}$. We define τ_S to be the topology over $\mathcal{B}_\mathcal{F}$ obtained by refining the canonical Baire topology of ω^ω restricted to $\mathcal{B}_\mathcal{F}$ by declaring the sets $C(\alpha)$ open.

Remark 3.4.31. It is straight forward that each $\alpha \in \omega_1$ has as a local base the following family:

$$\{C(\alpha) \cap [s] : s \in \omega^{<\omega} \textit{ and } f_\alpha \in [s]\}.$$

Here, $[s] = \{f \in \omega^\omega : s \subseteq f\}$ The following is based on Todorčević proof that $\mathfrak{b} = \omega_1$ implies that there is an S-space.

Proposition 3.4.32. $(\mathcal{B}_\mathcal{F}, \tau_S)$ *is an S-space.*

Proof. This proposition will be proved by appealing to the Lemma 3.4.28. For each $\alpha \in \omega_1$ we have that $C(\alpha)$ is closed in the Baire topology. Therefore, such set is clopen in τ_S. From this it easily follows that $\mathcal{B}_\mathcal{F}$ is 0-dimensional. Consequently, $\mathcal{B}_\mathcal{F}$ is also regular . Moreover, $\mathcal{B}_\mathcal{F}$ is left-open by definition. Thus, the only thing left to show is that $\mathcal{B}_\mathcal{F}$ does not contain any uncountable discrete set.

Let $S \in [\omega_1]^{\omega_1}$ and assume towards a contradiction that $\langle f_\alpha \rangle_{\alpha \in S}$ is discrete. In this way, for each $\alpha \in S$ we can find $s_\alpha \in \omega^{<\omega}$ so that, for $U_\alpha = C(\alpha) \cap [s_\alpha]$, we have that $U_\alpha \cap Y = \{\alpha\}$. Let $W \in [S]^{\omega_1}$ and $s \in \omega^{<\omega}$ for which $s_\alpha = s$ for all $\alpha \in W$. According to the Theorem 3.4.6, there are $\alpha < \beta \in W$ for which $osc(\alpha, \beta) = 0$. In other words, $f_\alpha < f_\beta$. By definition of $C(\beta)$, $f_\alpha \in C(\beta)$. Thus, $f_\alpha \in U_\beta$ which is a contradiction. $\blacksquare$

Corollary 3.4.33 (CA_2)**.** *There is a first countable S-space.*

Now, we present the construction of a distinct S-space. For this construction, we adapt the ideas from Chapter 2 of [121].

Definition 3.4.34. Given $\beta \in \omega_1$, we define $H(\beta) = \{\alpha < \beta : \rho(\alpha, \beta) = \Delta(\alpha, \beta)\}$.

Remark 3.4.35. Given $\beta \in \omega_1$, the set of all $\alpha < \beta$ so that $\{\alpha, \beta\}$ is captured, is contained in $H(\beta)$. Furthermore, if $\mathcal{F}$ is a morass then these two sets are equal.

As a consequence of the previous remark, and since $\mathcal{F}$ is 2-capturing, we also have the following.

Lemma 3.4.36. *Let $\mathcal{S} \subseteq FIN(\omega_1)$ be an uncountable family of pairwise disjoint sets. Then there are $a, b \in \mathcal{S}$ of the same cardinality n such that $a < b$ and $a(i) \in H(b(i))$ for all $i < n$.*

Definition 3.4.37. For each $\beta \in \omega_1$, we recursively define $C(\beta) \subseteq \beta + 1$ as the set containing β and all $\alpha < \beta$ for which there is $\gamma \in H(\beta)$ such that:

(a) $\alpha \in C(\gamma)$,

(b) for all $\gamma \neq \xi \in H(\beta) \cup \{\beta\}$, $\Delta(\alpha, \gamma) > \Delta(\alpha, \xi)$.

Finally, we define $C_k(\beta) = \{\alpha \in C(\beta) : \Delta(\alpha, \beta) \geqslant k\}$ for each $k \in \omega$.

Remark 3.4.38. Note that $C_l(\beta) \subseteq C_k(\beta)$ for all $k < l \in \omega$.

Remark 3.4.39. If $\gamma \in H(\beta)$, evidently $\gamma \in C(\gamma)$. Furthermore, if $\gamma \neq \xi \in H(\beta) \cup \{\beta\}$ then $\Delta(\gamma, \gamma) = \omega > \Delta(\gamma, \xi)$. From this we conclude that $H(\beta) \subseteq C(\gamma)$.

Lemma 3.4.40. *Let $\beta \in \omega_1$ and $\gamma \in H(\beta)$. Then $C_l(\gamma) \subseteq C(\beta)$ for $l = \Delta(\gamma, \beta) + 1$.*

Proof. Let $\alpha \in C_l(\gamma)$. According to the Definition 3.4.37, it suffices to show that $\Delta(\alpha, \xi) < \Delta(\alpha, \gamma)$ for each $\xi \in H(\beta) \cup \{\beta\}$ distinct from γ. If $\xi = \beta$ then $\Delta(\gamma, \beta) < l \leqslant \Delta(\alpha, \gamma)$. Therefore, $\Delta(\alpha, \beta) = \Delta(\gamma, \beta)$ due to the Lemma 2.3.4. In particular, $\Delta(\alpha, \beta) < \Delta(\alpha, \gamma)$. Now, if $\xi \in H(\beta)$ let us suppose towards a contradiction that $\Delta(\alpha, \gamma) \leqslant \Delta(\alpha, \xi)$. From the definition of l and the equality obtained in the case where $\xi = \beta$, we get that $\Delta(\alpha, \beta) = \Delta(\gamma, \beta) < l \leqslant \Delta(\alpha, \gamma) \leqslant \Delta(\alpha, \xi)$. Therefore,

$$\Delta(\beta, \xi) = \Delta(\alpha, \beta) = \Delta(\gamma, \beta)$$

by means of Lemma 2.3.4. The definition of $H(\beta)$ yields that

$$\Delta(\xi, \gamma) \leqslant \rho(\xi, \gamma) \leqslant \max(\rho(\xi, \beta), \rho(\gamma, \beta)) = \max(\Delta(\xi, \beta), \Delta(\gamma, \beta)) < \Delta(\alpha, \gamma).$$

In this way, $\Delta(\xi, \gamma) = \Delta(\alpha, \xi)$. This implies that $\Delta(\alpha, \xi) < \Delta(\alpha, \gamma)$ which is a contradiction. ∎

Lemma 3.4.41. *Let $\beta \in \omega_1$. For each $k \in \omega$ and $\alpha \in C_k(\beta)$ there is $l \in \omega$ such that $C_l(\alpha) \subseteq C(\beta)$.*

Proof. The proof is carried by induction over β. So suppose that we have proved the lemma for each ordinal smaller than some $\beta \in \omega_1$. Let $\alpha \in C_k(\beta)$. According to the Definition 3.4.37, there is $\gamma \in H(\beta)$ so that $\alpha \in C(\gamma)$ and $\Delta(\alpha, \gamma) > \Delta(\alpha, \xi)$ for each $\xi \in H(\beta) \cup \{\beta\}$. Particularly, $\Delta(\alpha, \gamma) > \Delta(\alpha, \beta)$. Lemma 2.3.4 implies that $\Delta(\alpha, \beta) = \Delta(\gamma, \beta)$. Now, by the Lemma 3.4.40, $C_{l_0}(\gamma) \subseteq C(\beta)$ where $l_0 = \Delta(\gamma, \beta) + 1$. Furthermore, $C_{l_1}(\alpha) \subseteq C_{\Delta(\alpha,\gamma)}(\gamma)$ for some $l_1 \in \omega$ due to the inductive hypotheses. Note that $C_{\Delta(\alpha,\gamma)}(\gamma) \subseteq C_{l_0}(\gamma)$. In this way, $C_{l_1}(\alpha) \subseteq C(\beta)$.

In order to finish, let $l = \max(\Delta(\alpha, \beta), l_1) + 1$. We claim that $C_l(\alpha) \subseteq C_k(\beta)$. For this purpose, take $\xi \in C_l(\alpha)$. Then $\xi \in C(\beta)$ because $l > l_1$. The only thing missing is to show that $\Delta(\xi, \beta) > k$. Indeed, $\Delta(\xi, \alpha) \geqslant l > \Delta(\alpha, \beta)$. Thus, by the Lemma 2.3.4, $\Delta(\xi, \beta) = \Delta(\alpha, \beta) > k$. ∎

Remark 3.4.42. By the previous corollary it is easy to see that the set $\{C_k(\beta) : k \in \omega \text{ and } \beta \in \omega_1\}$ forms a base for a topology in ω_1. It turns out that this defines a first countable locally compact strong S-space. It is convenient to transfer such topology to the family $\mathcal{B}_\mathcal{F}$.

Definition 3.4.43 (The topology τ_C)**.** Let $\beta \in \omega_1$ and $k \in \omega$. We define $\hat{C}_k(\beta) = \{f_\alpha \,|\, \alpha \in C_k(\beta)\}$ and $\hat{C}(\beta) = \{f_\alpha \,|\, \alpha \in C(\beta)\}$. Note that $\{\hat{C}_k(\beta) : k \in \omega \text{ and } \beta \in \omega_1\}$ forms a base for a topology over $\mathcal{B}_\mathcal{F}$. We will call this topology τ_C.

The following lemma follows directly from the fact that $\hat{C}_k(f_\beta) \subseteq [f_\beta|_k]$ for each $\beta \in \omega_1$ and $k \in \omega$.

Lemma 3.4.44. *Let $s \in \omega^{<\omega}$. Then $[s] \cap \mathcal{B}_\mathcal{F}$ is open in $(\mathcal{B}_\mathcal{F}, \tau_C)$. In particular, $(\mathcal{B}_\mathcal{F}, \tau_C)$ is Hausdorff.*

Lemma 3.4.45. *Let $\beta \in \omega_1$. $\hat{C}_k(\beta)$ is compact for each $k \in \omega$.*

Proof. The proof is carried by induction over β. Let $\langle \alpha_n \rangle_{n \in \omega} \subseteq C_k(\beta)$. We need to show that $S = \{f_{\alpha_n} : n \in \omega\}$ has an accumulation point in $\hat{C}_k(\beta)$. If the set of all $\Delta(\alpha_n, \beta)$ is unbounded in ω, clearly β is an accumulation point of S. So let us assume that such collection is bounded. Then there is $A \in [\omega]^\omega$ and $k \leqslant z \in \omega$ so that $\Delta(\alpha_n, \beta) = z$ for any $n \in A$. Given any such n, let $\gamma_n \in H(\beta)$ be so that $\alpha_n \in C(\gamma_n)$ and $\Delta(\alpha_n, \gamma_n) > \Delta(\alpha_n, \xi)$ for each $\xi \in H(\beta) \cup \{\beta\}$ distinct from γ_n. Particularly, we have that $\Delta(\alpha_n, \gamma_n) > \Delta(\alpha_n, \beta)$. Thus, $\Delta(\gamma_n, \beta) = \Delta(\alpha_n, \beta) = z$ due to the Lemma 2.3.4. As $\gamma_n \in H(\beta)$ then $\rho(\gamma_n, \beta) = z$. As $(\beta)_z$ is finite, there is $B \in [A]^\omega$ and $\gamma \in H(\beta)$ so that $\gamma_n = \gamma$ for all $n \in B$. In this way, $\alpha_n \in C(\gamma)$ and $\Delta(\alpha_n, \gamma) > \Delta(\alpha_n, \beta) = \Delta(\gamma, \beta)$ for each $n \in B$. In other words,

$$\langle \alpha_n \rangle_{n \in B} \subseteq C_l(\gamma)$$

where $l = \Delta(\gamma, \beta) + 1$. According to the inductive hypotheses, there is $\delta \in C_l(\gamma)$ for which f_δ is an accumulation point of $\langle \alpha_n \rangle_{n \in \beta}$. By means of Lemma 3.4.40, $\delta \in C(\beta)$. To finish, just note that $\Delta(\gamma, \beta) < l \leqslant \Delta(\gamma, \delta)$. Therefore, $\Delta(\beta, \delta) = \Delta(\gamma, \beta) = z \geqslant k$. Consequently, $\delta \in C_k(\beta)$. ∎

Proposition 3.4.46. *$(\mathcal{B}_\mathcal{F}, \tau_C)$ is a locally compact strong S-space.*

Proof. By definition, $\mathcal{B}_{\mathcal{F}}$ is left-open and it is locally compact due to Lemma 3.4.45. By Lemma 3.4.44, $\mathcal{B}_{\mathcal{F}}$ is also Hausdorff. The last two properties imply that the space is T_3. Fix $n \in \omega$. It remains to prove that $\mathcal{B}_{\mathcal{F}}^{n+1}$ has no uncountable discrete subspaces. For this, let $S \subseteq \omega_1^n$ be uncountable, and assume towards a contradiction that $\langle (f_{x(0)}, \ldots, f_{x(n)}) \rangle_{x \in S}$ is discrete. Without loss of generality we can suppose $x(i) < x(j)$ whenever $i < j \leqslant n$ and $\langle x(i) \rangle_{i \leqslant n} \cap \langle y(i) \rangle_{i \leqslant n} = \emptyset$ for all $x, y \in S$ with $x \neq y$. Furthermore, be a refining argument we can also suppose there is $k \in \omega$ such that for all $x, y \in S$, the following happens:

1. $\left(\prod\limits_{i \leqslant n} C_k(x(i)) \right) \cap S = \{x\}$,

2. $f_{x(i)}|_k = f_{y(i)}|_k$ for every $i \leqslant n$.

Due to Lemma 3.4.36, we know there are distinct $x, y \in S$ such that $x(i) \in H(y(i))$ for every $i \leqslant n$. For any such i, we know $\Delta(x(i), \xi) = \omega$ if and only if $\xi = x(i)$. Since $x(i)$ clearly belongs to $C(x(i))$, it follows from the definition that $x(i) \in C(y(i))$. But $f_{x(i)}|_k = f_y(i)|_k$, so $x(i)$ in fact is an element of $C_k(x(i))$. In this way, we conclude that $x \in \left(\prod\limits_{i \leqslant n} C_k(y(i)) \right) \cap S$, which is a contradiction. $\blacksquare$

Proposition 3.4.47. *The Alexandroff compactification of $(\mathcal{B}_{\mathcal{F}}, \tau_C)$ is a compact strong S-space.*

Corollary 3.4.48 (CA_2)**.** *There is a scattered compact strong S-space K whose function space $C(K)$ is hereditarily weakly Lindelöf and whose space $P(K)$ of all probability measures is also a strong S-space.*

3.4.4   Failure of Baumgartner's Axiom

As we saw on Theorem 3.2.33, FCA implies the existence of entangled sets, which means FCA also implies the failure of $BA(\omega_1)$. Although we do not know if CA_2 implies the existence of an entangled set, we will prove in this subsection that it does imply the negation of $BA(\omega_1)$. Our proof is based on Stevo Todorčević proof that $\mathfrak{b} = \omega_1$ implies the failure of $BA(\omega)$ (see page 308 of [103]). Remember that at the beginning of this section we fixed 2-capturing construction scheme, namely $\mathcal{F}$.

Lemma 3.4.49. *For any $\alpha \in \omega_1$ there are infinitely many $k \in \omega$ for which $0 \leqslant \Xi_\alpha(k) < n_k - 1$.*

Proof. Let $\alpha \in \omega_1$. Assume towards a contradiction that there is $k \in \omega$ so that for any $l > k$ we have that $\Xi_\alpha(l) = n_l - 1$. Since ρ is an unbounded metric, there are $\beta < \gamma \in \omega_1$ so that $\alpha < \beta$ and $l = \rho(\beta, \gamma) > k$. According to the Lemma 2.3.9 we have that $\Xi_\beta(l) \leqslant \Xi_\gamma(l) \leqslant n_l - 1$. Let $F \in \mathcal{F}_l$ be such that $\{\beta, \gamma\} \subseteq F$ and $a = \|\alpha\|_l$. Observe that since $\alpha < \beta$ and $\Xi_\beta(l) < n_l - 1$ then $F_{n_l-1} \backslash R(F) \subseteq \omega_1 \backslash (\alpha + 1)$. In this

way, $\alpha < F(a)$. This is because $\Xi_\alpha(l) = n_l - 1$. To finish, just note that $\|\alpha\|_l = \|F(a)\|_l$. This means that

$$\rho(\alpha, F(a)) \geqslant \Delta(\alpha, F(a)) \geqslant l > k.$$

By the point (b) of Lemma 2.3.9 we get that $0 \leqslant \Xi_\alpha(l') < \Xi_{F(a)}(l')$ where $l' = \rho(\alpha, F(a))$. This is a contradiction, so the proof is over. ∎

Now, let us fix M a countable elementary submodel of some large enough $H(\theta)$ with $\mathcal{F}, \mathcal{B}_{\mathcal{F}} \in M$. Define $\mathcal{A} = \mathcal{B}_{\mathcal{F}} \backslash M$. The following lemma is a direct consequence of elementarity.

Lemma 3.4.50. *Let $s \in \omega^{<\omega}$. If $\mathcal{A} \cap [s] \neq \emptyset$, then $\mathcal{A} \cap [s]$ is uncountable.*

Remember that, by Definition 3.2.31 and Remark 3.2.32, we can think of $(\mathcal{A}, <_{lex})$ as a subset of $\mathbb{R}$.

Lemma 3.4.51. $(\mathcal{A}, <_{lex})$ *is ω_1-dense.*

Proof. Let $f_\alpha, f_\beta \in \mathcal{A}$ with $f_\alpha <_{lex} f_\beta$, and let $l = \Delta(\alpha, \beta)$. According to the Lemma 3.4.49, there is $k > l$ so that $0 \leqslant \Xi_k(\alpha) < n_k - 1$. Consider $F \in \mathcal{F}_k$ such that $\alpha \in F$. Then $\alpha \in F \backslash F_{n_k - 1}$. Thus, the unique $\gamma \in F_{n_k - 1}$ for which $\|\gamma\|_{k-1} = \|\alpha\|_{k-1}$ is greater than α. Furthermore, $\|\gamma\|_k > \|\alpha\|_k$ and $f_\gamma \in \mathcal{A}$ because $f_\alpha \in \mathcal{A}$. In this way, $S = \mathcal{A} \cap [f_\gamma|_k]$ is uncountable due to the Lemma 3.4.50. In order to finish, we will show that S is contained in the open interval given by f_α and f_β. Indeed, let $f_\xi \in S$. By definition, $f_\xi|_k = f_\alpha|_k$. This implies that $f_\xi <_{lex} f_\beta$ because $k > l$. Finally, since $f_\alpha(k) = \|\alpha\|_k < \|\gamma\|_k = f_\gamma(k) = f_\xi(k)$ then $f_\alpha <_{lex} f_\xi$. ∎

Definition 3.4.52. Let $\alpha \in \omega_1$. We define $h_\alpha : \omega \longrightarrow \omega$ as:

$$h_\alpha(i) = m_i - f_\alpha(i.)$$

Additionally, we let $-\mathcal{A} = \{h_\alpha(i) \mid \alpha \in \omega_1 \backslash M\}$.

It is easy to see that $f_\alpha <_{lex} f_\beta$ if and only if $h_\beta <_{lex} h_\alpha$. Hence, we have the following corollary.

Corollary 3.4.53. $(-\mathcal{A}, <_{lex})$ *is ω_1-dense.*

Proposition 3.4.54. *There is no increasing function from $\mathcal{A}$ to $-\mathcal{A}$.*

Proof. Let us assume towards a contradiction that there is an increasing function $\Psi : \mathcal{A} \longrightarrow -\mathcal{A}$ increasing. Let $\psi : \omega_1 \backslash M \longrightarrow \omega_1 \backslash M$ be the unique function so that $\Psi(f_\alpha) = h_{\psi(\alpha)}$.

<u>Claim</u>: ψ has at most on fixed point.

Proof. Suppose that this is not true and let $\alpha, \beta \in \omega_1$ be distinct fixed points of ψ such that $f_\alpha <_{lex} f_\beta$. As Ψ is increasing, then

$$h_\alpha = h_{\psi(\alpha)} = \Psi(f_\alpha) <_{lex} \Psi(f_\beta) = h_{\psi(\beta)} = h_\beta.$$

But this is a contradiction since, in fact, $h_\beta <_{lex} h_\alpha$. □

Now, let $X = \{\alpha \in \omega_1 : \alpha < \psi(\alpha)\}$ and $Y = \{\alpha \in \omega_1 : \alpha > \psi(\alpha)\}$. By the previous claim, one of this sets is uncountable. Suppose without loss of generality that X is uncountable. For each $\alpha \in X$, let $b_\alpha = \{\alpha, \phi(\alpha)\}$. Since $\mathcal{F}$ is 2-capturing, we can find distinct $\alpha, \beta \in X$ for which $\{b_\alpha, b_\beta\}$ is captured. Observe that $f_\alpha <_{lex} f_\beta$ and $f_{\psi(\alpha)} <_{lex} f_{\psi(\beta)}$, or equivalently, $\Psi(f_\alpha) = h_{\psi(\alpha)} >_{lex} h_{\psi(\beta)} = \Psi(f_\beta)$. Note that this is a contradiction. Thus, the proof is over. ∎

Corollary 3.4.55 (CA_2)**.** *There are two ω_1-dense sets of reals which are not isomorphic.*

3.5 On σ-monotone spaces

Monotone and σ-monotone spaces are a particular kind of metric spaces which were defined in [83] by Aleš Nekvinda and Ondřej Zindulka. In [135], Zindulka used such spaces to prove the existence of universal measure zero sets of large Hausdorff dimension.

Definition 3.5.1. Let (X, d) be a metric space and $c > 0$. We say that X is *c-monotone* if there is a linear order $<$ on X such that for all $x < y < z \in X$, the following inequality holds:
$$d(x, y) \leqslant c \cdot d(x, z).$$
We say that X is *monotone* if it is c-monotone for some $c > 0$. Finally, we say that X is *σ-monotone* if it is a countable union of monotone subspaces.

In [51], Zindulka and Michael Hrušák investigated the (possibly trivial) ideal $Mon(X)$ of all σ-monotone subspaces of a given metric space (X, d). Particularly, they showed that every separable metric space of size less than the cardinal invariant $\mathfrak{m}_{\sigma-linked}$ is σ-monotone. In other words, for all such spaces X, the ideal $Mon(X)$ is trivial. In Question 6.7 of that same paper, they asked whether the previous result remains true for non-separable spaces. That is:

Is there a metric space of cardinality ω_1 that is not σ-monotone?

In this section we will show that the capturing axiom CA implies the existence of a metric space of cardinality ω_1 which has no uncountable σ-monotone subspaces. Therefore, it is consistent with arbitrarily large values of the continuum that the previous question has an affirmative answer.

Remark 3.5.2. Let $r, c > 0$ and (Y, d) be a c-monotone space. Then $(Y, r \cdot d)$ is also c-monotone.

Remark 3.5.3. Let $c > 0$ and (Y, d) be a c-monotone space. If $Z \subseteq Y$ then Z is also c-monotone.

Remark 3.5.4. Let $c > 0$ and (Y, d) be a c-monotone space. Then Y is c'-monotone for each $c' > c$.

By the previous remark, it is easy to see that a space Y is not monotone if and only if it is not $\frac{1}{n}$ for each $0 < n \in \omega$. The following result is implicit in Lemma 4.4 and Proposition 4.5 of [83].

Lemma 3.5.5. *Let $n \in \omega$. There is a finite metric space (Z_n, d) so that Z_n is not $\frac{1}{n}$-monotone.*

Remark 3.5.6. Note that if m is a natural number for which there is a space Z_n of size m which is not $\frac{1}{n}$-monotone, then the same is true for each $m' \geqslant m$.

Theorem 3.5.7 (Under CA). *There is a metric d over ω_1 so that (ω_1, d) has no uncountable monotone subspaces.*

Proof. Let $\tau = \langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$ be a type so that for any $k > 0$, there is a metric d_k over n_k for which the following conditions holds:

For each $0 < k' \leqslant k$, $n_{k'}$ is not a $\frac{1}{k'}$-monotone subspace of (n_k, d_k).

Such type exists due to the Lemma 3.5.5 and the Remarks 3.5.3 and 3.5.6. Furthermore, by Remark 3.5.2 we may assume that $diam(n_k) = 1$ for each k. Now, let $\mathcal{F}$ be a capturing construction scheme of type τ. We will proceed to define for each $F \in \mathcal{F}$ a metric d_F over F in such way that the following properties hold for any two $F, G \in \mathcal{F}$:

(a) If $\rho^F = \rho^G$ and $h : F \longrightarrow G$ is the increasing bijection, then h is also an isometry between (F, d_F) and (G, d_F).

(b) If $F \subseteq G$, then (F, d_F) is subspace of (G, d_F).

(c) If $k = \rho^F > 0$, there is $0 < s \in \mathbb{R}$ so that for any two $j, l < n_k$ and each $r_k \leqslant i < m_{k-1}$, $d_F(F_j(i), F_l(i)) = s \cdot d_k(j, l)$.

The proof is carried by recursion over $k = \rho^F$.

Base step: If $k = 1$, then $n_k = m_k = |F|$. In this case, we let $d_F : F^2 \longrightarrow \mathbb{R}$ be defined as:
$$d_F(F(i), F(j)) = d_1(i, j)$$
for all $i, j < F$.

Recursive step: Let $k \leqslant 1$ and suppose that we have defined, for all $1 \leqslant k' \leqslant k$, metrics d_G over each $G \in \mathcal{F}_{k'}$ in such a way that the properties (a), (b) and (c) hold. Given $G \in \mathcal{F}_k$, let
$$s_k^G = \min(d_G(\alpha, \beta) : \alpha \neq \beta \in G).$$
By means of the property (a), the number s_k^G does not depend on depend on G, so let just call s_k. Now, let $F \in \mathcal{F}_{k+1}$. Note that each element of F is of the form $F_j(i)$ for some $j < n_{k+1}$ and $i < m_k$. Having this in mind, we define $d_F : F^2 \longrightarrow F$ as:
$$d_F(F_j(i), F_l(t)) = \begin{cases} d_{F_j}(F_j(i), F_j(t)) & \text{if } i \neq t \\ \frac{s_k}{2} \cdot d_{k+1}(j, l) & \text{if } i = t \text{ and } F_j(i) \neq F_l(t) \\ 0 & \text{if } F_j(i) = F_l(t) \end{cases}$$

It is not hard to see that d_F is well defined and is, in fact, a metric. Furthermore, from the definition, it should be clear that for any $F, G \in \mathcal{F}$ with $\rho^F, \rho^G \leqslant k+1$, the properties (a), (b) and (c) are still true. This finishes the recursion.

Now, we define $d = \bigcup_{F \in \mathcal{F}} d_F$. By the property (b), it follows that d is a (well defined) metric over ω_1. We will now show that (ω_1, d) has no uncountable monotone subspaces. For this, let $S \in [\omega_1]^{\omega_1}$ According to the Remark 3.5.4, it is enough to show that for each $0 < k \in \omega$, S is not $\frac{1}{k}$-monotone. Indeed, since $\mathcal{F}$ is capturing (and thus, it is n_k-capturing) there is $D \in [S]^{n_k}$ which is captured at some level $l > k$. Let $F \in \mathcal{F}_l$ be such that $D \subseteq F$. Then $D = \{F(i) : i < n_k\}$ for some $r_l \leqslant i < m_{l-1}$. In this way, D is isomorphic to n_k seen as a subspace of $(n_l, \frac{s_l}{2} \cdot d_l)$. This is due to the property (c). But n_k is not an $\frac{1}{k}$-monotone subspace of n_l. We conclude, using the Remark 3.5.2, that D is not $\frac{1}{k}$-monotone. As $D \subseteq S$, the same holds for S. Thus, the proof is over.

$\blacksquare$

Chapter 4

Fragments of Martin's axiom

In this chapter we will study the relation between construction schemes and forcing notions which preserve some of their "capturing properties". Given an n-capturing construction scheme $\mathcal{F}$, we show that there is a filter $\mathcal{U}_n(\mathcal{F})$ over ω so that, for any partition $\mathcal{P}$, $\mathcal{F}$ is $\mathcal{P}$-n-capturing if and only if $\mathcal{P} \subseteq \mathcal{U}_n(\mathcal{F})^+$. Later, we will establish the consistency of the statement $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$ and prove that under this assumption, $\mathcal{U}_n(\mathcal{F})$ is an ultrafilter. In particular, this implies that there may by construction schemes which are n-capturing but not $\mathcal{P}$-n-capturing for any non-trivial partition. Finally, we will show that $\mathcal{U}_n(\mathcal{F})$ is in fact a Ramsey ultrafilter and give an example of an ultrafilter over ω_1 which can be explicitly defined from $\mathcal{F}$, and which can be projected to another Ramsey ultrafilter over ω.

Before stating and proving the results that are due to the author, it will be convenient to recall some of the previous work done by Damjan Kalajdzievski and Fulgencio Lopez regarding the preservation of capturing properties.

Definition 4.0.1 (*n-preserving property*)**.** Let $\mathbb{P}$ be a forcing notion and which preserves ω_1 and $\mathcal{F}$ be a construction scheme. Given $2 \leqslant n \in \omega$, we say that $\mathbb{P}$ *n-preserves* $\mathcal{F}$ if

$$\mathbb{P} \Vdash \text{`` } \mathcal{F} \text{ is } n\text{-capturing ''}.$$

Remark 4.0.2. If $\mathbb{P}$ and $\mathcal{F}$ are as in the previous definition, then $\mathcal{F}$ is n-capturing.

Usually, proving that a *ccc* forcing $\mathbb{P}$ n-preserves a scheme $\mathcal{F}$, is similar to proving that $\mathbb{P}$ is *ccc*. This can be exemplified in the proof of Theorem 3.1.36. The following lemma gives us a useful way of handling this situation.

Lemma 4.0.3. *Let $\mathcal{F}$ be an n-capturing construction scheme and $\mathbb{P}$ be a forcing. The two following statements are equivalent:*

(1) *$\mathbb{P}$ is ccc and n-preserves $\mathcal{F}$.*

(2) *For any $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$ and each injective function $\nu : \mathcal{A} \longrightarrow \omega_1$, there is $\{p_0, \ldots, p_{n-1}\} \in [\mathcal{A}]^n$ for which:*

- *$\{p_0, \ldots, p_{n-1}\}$ is centered. That is, there is $p \in \mathbb{P}$ so that $p \leqslant p_i$ for any $i < n$.*

- $\{\nu(p_0), \dots, \nu(p_{n-1})\}$ *is captured.*

Proof. Proof of $\Rightarrow$. Let $\mathcal{A}$ be an uncountable subset of $\mathbb{P}$ and consider $\nu : \mathcal{A} \longrightarrow \omega_1$ an injective function. As $\mathbb{P}$ is *ccc*, there is G a $\mathbb{P}$-generic filter over V for which $G \cap \mathcal{A}$ is uncountable. In $V[G]$, let $X = \nu[\mathcal{A} \cap G]$. Note that X is uncountable since ν is injective. As $\mathbb{P}$ n-preserves $\mathcal{F}$, there is $\{\alpha_0, \dots, \alpha_{n-1}\} \in [X]^n$ which is captured. Given $i < n$, let $p_i \in \mathcal{A} \cap G$ be such that $\nu(p_i) = \alpha_i$. Then $\{p_0, \dots, p_{n-1}\}$ is centered because it is included in $\mathcal{A}$, and $\{\nu(p_0), \dots, \nu(p_{n-1})\}$ is captured. $\qquad\square$

Proof of $\Leftarrow$. In order to show that $\mathbb{P}$ is *ccc* let $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$. Let us consider $\nu : \mathcal{A} \longrightarrow \omega_1$ an aribtrary inyective function. According to the hypotheses, there is $\{p_0, \dots p_{n-1}\} \in [\mathcal{A}]^n$ which is centered. Particularly, p_0 and p_1 are two distinct compatible elements of $\mathcal{A}$. Thus, $\mathcal{A}$ is not an antichain.

Now we will show that $\mathbb{P}$ n-preserves $\mathcal{F}$. For this purpose, let G be a $\mathbb{P}$-generic filter over V and $X \in V[G] \cap [\omega_1]^{\omega_1}$. Let $\dot{X}$ be a name for X which is forced to be an uncountable subset of ω_1 by $\mathbb{1}_{\mathbb{P}}$. To finish, it is enough to show that the set of all $p \in \mathbb{P}$ for which there is $D \in [\omega_1]^n$ so that D is captured and $p \Vdash$ " $D \subseteq \dot{X}$", is dense in $\mathbb{P}$. For this purpose, let $q \in \mathbb{P}$. If there is $p \leqslant q$ for which the set $Y_p = \{\alpha \in \omega_1 : p \Vdash \text{"}\alpha \in \dot{X}\text{"} \}$ is uncountable, we are done. Therefore, we may assume that Y_p is at most countable for each $p \leqslant q$. From this fact, it is easy to see that there is $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$ and an injective function $\nu : \mathcal{A} \longrightarrow \omega_1$ so that $p \leqslant q$ and $p \Vdash$ " $\nu(p) \in \dot{X}$" for any $p \in \mathcal{A}$. According to the hypotheses, there is $\{p_0, \dots, p_{n-1}\} \in [\mathcal{A}]$ which is centered and for which $D = \nu[\{p_0, \dots, p_{n-1}\}]$ is captured. Let p be such that $p \leqslant p_i$ for each $i < n$. Then $p \leqslant q$ and $p \Vdash \text{"}D \subseteq \dot{X}\text{"}$. This finishes the proof. $\qquad\square$

$\qquad\blacksquare$

The following lemma was proved in [58].

Lemma 4.0.4. *Let $n \in \omega$, $\mathcal{P}$ a partition of ω and suppose that $\mathcal{F}$ is a $\mathcal{P}$-n-capturing construction scheme. If $\mathbb{P}$ is an n-Knaster forcing then $\mathbb{P}$ $\mathcal{P}$-n-preserves $\mathcal{F}$.*

Proof. Let $\dot{X}$ be a name for an uncountable subset of ω_1. We will prove the lemma by appealing to the equivalence of n-capturing provided by Lemma 2.4.15. Consider an arbitrary $q \in \mathbb{P}$. Let $\langle p_\xi, \alpha_\xi \rangle_{\xi \in \omega_1} \subseteq \mathbb{P} \times \omega_1$ be a sequence so that the following properties hold:

- For all $\xi \in \omega_1$, $p_\xi \leqslant q$ and $p_\xi \Vdash$ " $\alpha_\xi \in \dot{X}$ ".

- For any two distinct $\xi, \mu \in \omega_1$, $\alpha_\xi \neq \alpha_\mu$.

Since $\mathbb{P}$ is n-Knaster, there is $S \in [\omega_1]^{\omega_1}$ such that $\{p_\xi : \xi \in S\}$ is n-linked. Now, as $\mathcal{F}$ is $\mathcal{P}$-n-capturing and S is uncountable, for any $P \in \mathcal{P}$ there are distinct $\xi_0, \dots, \xi_n \in S$ for which $D = \{\alpha_{\xi_0}, \dots, \alpha_{\xi_{n-1}}\}$ is captured at some level $l \in P$. Let $p \in \mathbb{P}$ be such that $p \leqslant p_{\xi_i}$ for each $i < n$. Then $p \leqslant q$ and $p \Vdash$ " $D \in [\dot{X}]^n$ and is captured at l ". This finishes the proof. $\qquad\blacksquare$

In [58], Damjan Kalajdzievski and Fulgencio Lopez used the previous lemma to show that $MA(K_n)$ is consistent with CA_n.

Theorem 4.0.5. *Let $2 \leqslant n \in \omega$ and V be a model of CH. For any cardinal $\kappa > \omega_1$ of uncountable cofinality, there is an n-Knaster forcing $\mathbb{K}$ so that*

$$\mathbb{K} \Vdash \text{``} \mathfrak{c} = \kappa + MA(K_n) + CA_n \text{''}.$$

Proof. Let $(\langle \mathbb{K}_\xi \rangle_{\xi \leqslant \kappa}, \langle \dot{\mathbb{Q}}_\xi \rangle_{\xi < \kappa})$ be a finite support iteration of n-Knaster forcings of length κ. If an appropriate bookkeeping is used, we can summon both Lemma 4.0.4 and Theorem 2.4.16 to show that $\mathbb{K} = \mathbb{K}_\kappa$ satisfies the desired conclusions. ∎

In that same paper, they showed that $\mathfrak{m}_{K_n} > \omega_1$ implies that there are no $n + 1$-capturing construction schemes. For that purpose, they used the following property.

The Property $(\star)_n$: For any uncountable $\Gamma \subseteq \omega^\omega$ there is $\Gamma_0 \in [\Gamma]^{\omega_1}$ such that there are no $g_0, \ldots, g_n \in \Gamma_0$ and $k \in \omega$ with $g_0|_k = \cdots = g_n|_k$ and $g_i(k) \neq g_j(k)$ for any $i < j \leqslant n$.

The property $(\star)_n$ was considered by Stevo Todorčević in [119]. The following theorem follows from Lemma 6 of such paper.

Theorem 4.0.6. *Let $2 \leqslant n \in \omega$. Then $\mathfrak{m}_{K_n} > \omega_1$ implies $(\star)_n$.*

In contrast, $n + 1$-capturing construction schemes imply the failure of $(\star)_n$ (see Theorem 2.4 in [58]).

Theorem 4.0.7. *Let $1 \leqslant n \in \omega$. If there is an $n + 1$-capturing construction scheme, then $(\star)_n$ fails.*

Proof. Let $\mathcal{F}$ be an $n+1$-capturing construction scheme. For any $\alpha \in \omega_1$ let $f_\alpha : \omega \longrightarrow \omega$ be defined as:

$$f_\alpha(k) = \|\alpha\|_k.$$

Now, let $\Gamma = \{ f_\alpha : \alpha \in \omega_1 \}$. Since $\mathcal{F}$ is $n + 1$-capturing, for any $S \in [\omega_1]^{\omega_1}$ there is $D \in [S]^{n+1}$ which is captured at some level l. Then $g_{D(0)}|_l = \cdots = g_{D(n)}|_l$ and $g_{D(i)}(l) < g_{D(j)}(l)$ for any $i < j \leqslant n$. In this way, Γ testifies the failure of $(*)_n$. ∎

4.1 The principle $\mathfrak{m}_{\mathcal{F}}^n$

The main goal of this small section is to show that for a given n-capturing construction scheme $\mathcal{F}$, it is consistent that $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$. Let us first recall the exact definition of $\mathfrak{m}_{\mathcal{F}}^n$.

Definition 4.1.1. Let $\mathcal{F}$ be a construction scheme and $n \in \omega$. We define $\mathfrak{m}_{\mathcal{F}}^n$ as follows:

$$\mathfrak{m}_{\mathcal{F}}^n = \begin{cases} \omega & \text{if } \mathcal{F} \text{ is not } n\text{-capturing} \\ \min(\mathfrak{m}(\mathbb{P}) \, : \, \mathbb{P} \text{ is ccc and } \mathbb{P} \text{ } n\text{-preserves } \mathcal{F}) & \text{if } \mathcal{F} \text{ is } n\text{-capturing} \end{cases}$$

$\mathfrak{m}_{\mathcal{F}}^2$ is denoted simply as $\mathfrak{m}_{\mathcal{F}}$.

Remark 4.1.2. Suppose that $\mathcal{F}$ is an n-capturing construction scheme. The following diagram represents the basic relations between the cardinals $\mathfrak{m}_{\mathcal{F}}^i$ and $\mathfrak{m}_{K_i}$ for each $i \leqslant n$. Note that the non-trivial relations hold due to Lemma 4.0.4.

$$
\begin{array}{ccccccccc}
\mathfrak{m}_{K_2} & \longleftarrow & \mathfrak{m}_{K_3} & \longleftarrow & \cdots & \longleftarrow & \mathfrak{m}_{K_n} & \longleftarrow & \mathfrak{c} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\omega_1 & \longleftarrow & \mathfrak{m}_{\mathcal{F}}^2 & \longleftarrow & \mathfrak{m}_{\mathcal{F}}^3 & \longleftarrow & \cdots & \longleftarrow & \mathfrak{m}_{\mathcal{F}}^n
\end{array}
$$

As the following proposition suggests, having n-Knaster property may be optimal in terms n-preserving a construction scheme.

Proposition 4.1.3. *Let* $2 \leqslant n \in \omega$ *and* $\mathcal{F}$ *be an* n-capturing construction scheme *for which* $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$. *Given a ccc forcing* $\mathbb{P}$ *and* $m \geqslant n$, *the following statements are equivalent:*

(a) $\mathbb{P}$ *is* m-Knaster.

(b) $\mathbb{P}$ *is* n-Knaster.

(c) $\mathbb{P}$ n-preserves $\mathcal{F}$.

(d) $\mathbb{P}$ *has precaliber* ω_1.

Proof. (a) implies (b) and (d) implies (a) are obvious, and (b) implies (c) is just Lemma 4.0.4 (which does not require $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$). In order to prove that (c) implies (d), let $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$. Since $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$, so is $\mathfrak{m}(\mathbb{P})$. As $\mathbb{P}$ is *ccc*, this means that there is a filter G over $\mathbb{P}$ so that $\mathcal{A} \cap G$ is uncountable. $\blacksquare$

Corollary 4.1.4. *Let* $2 \leqslant n \in \omega$ *and* $\mathcal{F}$ *be an* n-capturing construction scheme. *If* $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$, *then* $\mathfrak{m}_{\mathcal{F}}^n = \mathfrak{m}_{K_m}$ *for each* $m \geqslant n$.

A a corollary of Theorem 4.0.7, we also have the following.

Corollary 4.1.5. *Let* $2 \leqslant n \in \omega$ *and* $\mathcal{F}$ *be an* n-capturing construction scheme. *If* $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$ *then* $\mathfrak{m}_{K_m} = \omega_1$ *for each* $m < n$ *and* $\mathfrak{m}_{\mathcal{F}}^m = \omega$ *for each* $m > n$.

In Corollary 4.3.4 we will explicitly show that the inequality $\mathfrak{m} > \omega_1$ is inconsistent with the existence of a 2-capturing construction scheme. As a consequence of such result, we have that the cardinals $\mathfrak{m}_{\mathcal{F}}^n$ and $\mathfrak{m}$ are incomparable in a strong sense.

Corollary 4.1.6. *Let* $2 \leqslant n \in \omega$ *and* $\mathcal{F}$ *be a construction scheme. Either* $\mathfrak{m}_{\mathcal{F}}^n \leqslant \omega_1$ *or* $\mathfrak{m} = \omega_1$.

The following Lemma is all that we need to prove the main result of this section.

Lemma 4.1.7. *Let* $\mathcal{F}$ *be an* n-capturing construction scheme *and* $(\langle \mathbb{P}_\xi \rangle_{\xi \leqslant \gamma}, \langle \dot{\mathbb{Q}}_\xi \rangle_{\xi < \gamma})$ *be a finite support iteration of ccc forcings so that*

$$
\mathbb{P}_\xi \Vdash \text{``} \dot{\mathbb{Q}}_\xi \ n\text{-preserves } \mathcal{F} \text{ ''}.
$$

Then $\mathbb{P}_\gamma$ *also* n-preserves $\mathcal{F}$.

Proof. The proof is carried by induction over γ by appealing to the equivalence of n-capturing provided by Lemma 2.4.15. Both the base and the successor steps of the induction are trivial to show. Hence, we will only do the limit case. For this, let us assume that γ is limit and we have already showed that $\mathbb{P}_\alpha$ n-preserves $\mathcal{F}$ for each $\alpha < \gamma$. Let $\dot{X}$ be a $\mathbb{P}_\gamma$-name for an uncountable subset of ω_1 and consider an arbitrary $q \in \mathbb{P}_\gamma$. We can take a sequence $\langle p_\xi, \alpha_\xi \rangle_{\xi \in \omega_1} \subseteq \mathbb{P}_\gamma \times \omega_1$ such that for any two distinct $\xi, \mu \in \omega_1$, the following properties hold:

- $p_\xi \leqslant q$ and $p_\xi \Vdash$ "$\alpha_\xi \in \dot{X}$ ".

- $\alpha_\xi \neq \alpha_\mu$.

By refining the sequence if necessary, we may assume that $\{dom(p_\xi) : \xi \in \omega_1\}$ forms a Δ-system with root $R \subseteq \gamma$. Now, let $\alpha < \gamma$ be such that $R \subseteq \alpha$. Observe that $q \in \mathbb{P}_\alpha$. As $\mathbb{P}_\alpha$ is *ccc*, there is G a $\mathbb{P}_\alpha$-generic filter over V so that $S = \{\xi \in \omega_1 : p_\xi|_\alpha \in G\}$ is uncountable. In particular, this implies that for any $\xi, \mu \in S$, the conditions $p_\xi|_\alpha$ and $p_\mu|_\alpha$ are compatible. Since $R = dom(p_\xi) \cap dom(p_\mu) \subseteq \alpha$, it follows that p_ξ and p_μ are compatible. Now, according to the inductive hypotheses, $\mathcal{F}$ is n-capturing inside $V[G]$. Therefore, there is $D \in [S]^{n-1}$ so that the family $\{\alpha_\xi : \xi \in D\}$ is captured. To finish, let $q \in \mathbb{P}_\gamma$ be such that $p \leqslant p_\xi$ for any $\xi \in D$. It is straightforward that $p \leqslant q$ and

$$p \Vdash \text{ "} \{\alpha_\xi : \xi \in D\} \in [\dot{X}]^n \text{ and it is captured ".}$$

Thus, the proof is over. ∎

Theorem 4.1.8. *Let $\kappa > \omega_1$ be a regular cardinal such that $2^{<\kappa} = \kappa$. Given $\mathcal{F}$ an n-capturing construction scheme, there is a ccc-forcing $\mathbb{P}$ for which*

$$\mathbb{P} \Vdash \text{ "} \mathfrak{m}_{\mathcal{F}}^n = \mathfrak{c} = \kappa \text{ ".}$$

Proof. We can construct a finite support iteration $(\langle \mathbb{P}_\xi \rangle_{\xi \leqslant \kappa}, \langle \dot{\mathbb{Q}}_\xi \rangle_{\xi < \kappa})$ of *ccc* forcings so that following properties hold for each $\xi < \kappa$:

- $\mathbb{P}_\xi \Vdash$ "$|\dot{\mathbb{Q}}_\xi| < \kappa$",

- $\mathbb{P}_\xi \Vdash$ " $\dot{\mathbb{Q}}_\xi$ n-preserves $\mathcal{F}$ ".

If the iteration is constructed with an appropriate bookkeeping, we can arrange that whenever G is $\mathbb{P}_\kappa$-generic over V and $\mathbb{Q}$ is a *ccc* forcing inside $V[G]$ of size less than κ which n-preserves $\mathcal{F}$, then there are cofinally many $\xi < \kappa$ for which $\mathbb{Q}$ is order isomorphic to $\dot{\mathbb{Q}}_\kappa^G$. By standard arguments, all of these properties imply that $\mathbb{P} = \mathbb{P}_\kappa$ satisfies the desired conclusion. ∎

4.2   The n-projection filter

Our next goal is to show construction schemes which are n-capturing may not be $\mathcal{P}$-n-capturing for any non-trivial partition $\mathcal{P}$ of ω. For this, we introduce a natural filter over ω which is definable from a construction scheme.

Definition 4.2.1. Let $2 \leqslant n \in \omega$ and $\mathcal{F}$ be a construction scheme. Given $S \subseteq \omega_1$, we define the *n-projection* of S as

$$\pi_n(S) = \{\rho^D \, : \, D \in [S]^n \text{ and } D \text{ is captured }\}$$

Remark 4.2.2. Given $S \subseteq \omega_1$, the n-projection of S can be recovered from any cofinal subset of $[S]^{<\omega}$. That is, if G is a cofinal subset of $[S]^{<\omega}$, then

$$\pi_n(S) = \bigcup_{D \in G} \pi_n(D).$$

Definition 4.2.3 (The n-projection filter)**.** Let $n \in \omega$ and $\mathcal{F}$ be a construction scheme. We define $\mathcal{U}_n(\mathcal{F})$ as the set of all $A \subseteq \omega$ for which there is $S \in [\omega_1]^{\omega_1}$ such that

$$\pi_n(S) \subseteq A.$$

Remark 4.2.4. Note that if $\mathcal{F}$ is not n-capturing then $\emptyset \in \mathcal{U}_n(\mathcal{F})$. Therefore, $\mathcal{U}_n(\mathcal{F})$ is not actually a filter in this case.

Lemma 4.2.5. *Let $2 \leqslant n \in \omega$ and $\mathcal{F}$ be a construction scheme. If $\mathcal{F}$ is n-capturing, then $\mathcal{U}_n(\mathcal{F})$ is a non-principal filter over ω.*

Proof. Since $\mathcal{F}$ is n-capturing, it follows that each member of $\mathcal{U}_n(\mathcal{F})$ is non-empty. Thus, in order to prove that $\mathcal{U}_n(\mathcal{F})$ is a filter, it is enough to show that the family $\{\pi_n(S) \, : \, S \in [\omega_1]^{\omega_1}\}$ is downwards directed with respect to $\subseteq$. Let $S, S' \in [\omega_1]^{\omega_1}$. We will prove that there is $A \in [\omega_1]^{\omega_1}$ such that $\pi_n(A) \subseteq \pi_n(S) \cap \pi_n(S')$. For this, first note that we can recursively construct a sequence $\langle \alpha_\xi, \beta_\xi \rangle \subseteq S \times S'$ so that $\alpha_\xi < \beta_\xi < \alpha_{\xi+1}$ for each $\xi \in \omega_1$. By refining the sequence if necessary, we may assume without loss of generality that there is $k \in \omega$ so that for any two distinct $\xi, \mu \in \omega_1$, the following conditions hold:

(1) $\rho(\alpha_\xi, \beta_\xi) < k$,

(2) $\|\alpha_\xi\|_k = \|\alpha_\mu\|_k$ and $\|\beta_\xi\|_k = \|\beta_\mu\|_k$. In particular, $\rho(\beta_\mu, \beta_\xi) > k$ by means of Lemma 2.3.3.

Let $A = \{\beta_\xi \, : \, \xi \in \omega_1\}$. Then $A \in [S']^{\omega_1}$, so $\pi_n(A) \subseteq \pi_n(S')$. We claim that $\pi_n(A) \subseteq \pi_n(S)$. For this purpose, let $l \in \pi_n(A)$ and consider $D \in [\omega_1]^n$ so that $\{\beta_\xi \, : \xi \in D\}$ is captured at level l. Note that $l > k$ due to the condition (2).

<u>Claim:</u> If $\xi, \mu \in D$ are distinct then $\Delta(\alpha_\xi, \alpha_\mu) = l = \rho(\alpha_\xi, \alpha_\mu)$.

Proof of claim. Let $h : (\beta_\xi)_k \longrightarrow (\beta_\mu)_k$ be the increasing bijection. By the point (1), we have that $\alpha_\xi \in (\beta_\xi)_k$ and $\alpha_\mu \in (\beta_\mu)_k$. Furthermore, $h(\alpha_\xi) = \alpha_\mu$ by the point (2). Since $\alpha_\xi \neq \alpha_\mu$, we may use Lemma 2.3.7 to conclude that

$$l = \rho(\beta_\xi, \beta_\mu) \geqslant \rho(\alpha_\xi, \alpha_\mu) \geqslant \Delta(\alpha_\xi, \alpha_\mu) \geqslant \Delta(\beta_\xi, \beta_\mu) = l.$$

So we are done. $\qquad\square$

Note that $\Xi_{\alpha_\xi}(l) \geqslant 0$ for each $\xi \in D$. This is due to the point (b) of Lemma 2.3.9. By the point (c) of such lemma, $\Xi_{\alpha_\xi}(l) = \Xi_{\beta_\xi}(l)$. Thus, $\{\alpha_\xi : \xi \in D\}$ is captured at level l by means of Proposition 2.4.10. That is, $l \in \pi_n(S)$. As $l \in \pi_n(A)$ was arbitrary, we get that $\pi_n(A) \subseteq \pi_n(S)$.

Now, we will show that $\mathcal{U}_n(\mathcal{F})$ is non-principal. Let $S \in [\omega_1]^{\omega_1}$ and $k \in \omega$. Then there is $S' \in [S]^{\omega_1}$ such that $\|\alpha\|_k = \|\beta\|_k$ for all $\alpha, \beta \in S$. This implies that $\rho^D > k$ for any $D \in [S']^n$. Therefore, $\pi_n[S'] \subseteq \pi_n[S]\backslash k$. This finishes the proof. ∎

Remark 4.2.6. Suppose that $\mathcal{F}$ is an *n*-capturing construction scheme. Note that $A \in \mathcal{U}_n(\mathcal{F})^+$ if and only if for any $S \in [\omega_1]^{\omega_1}$ there are infinitely many $l \in A$ for which there is $D \in [S]^n$ which is captured at level l. In this way, $\mathcal{F}$ is $\mathcal{P}$-*n*-capturing for a partition $\mathcal{P}$ of ω if and only if $\mathcal{P} \subseteq \mathcal{U}_n(\mathcal{F})^+$.

As a direct corollary of the previous remark we have that:

Corollary 4.2.7. *Let* $2 \leqslant n \in \omega$ *and* $\mathcal{F}$ *be a n-capturing construction scheme. There is a non-trivial partition* $\mathcal{P}$ *of* ω *for which* $\mathcal{F}$ *is* $\mathcal{P}$-*n-capturing if and only if* $\mathcal{U}_n(\mathcal{F})$ *is not an ultrafilter.*

4.3 *n*-capturing with partitions

In this subsection we will prove Theorem 2.4.18. Furthermore, we will show that it is consistent the existence of a construction scheme which is *n*-capturing and $\mathcal{P}$-$(n-1)$-capturing for some non-trivial partition of ω, but it is not $\mathcal{P}'$-*n*-capturing for any non-trivial partition $\mathcal{P}'$. This will be done by starting with a $\mathcal{P}$-*n*-capturing construction scheme, and building a suitable finite support iteration of *ccc* forcings which force $\mathcal{U}_n(\mathcal{F})$ to be an ultrafilter. In the following definition, we describe the forcings that will be used for this task.

Definition 4.3.1. Let $\mathcal{F}$ be a construction scheme. Given $2 \leqslant n \in \omega$ and $A \subseteq \omega$, we define the forcing $\mathbb{D}_n(\mathcal{F}, A)$ as the family of all $p \in [\omega_1]^{<\omega}$ with the following property:

There is no $D \in [p]^n$ such that D is captured at some level $l \in A$.

We order $\mathbb{D}_n(\mathcal{F}, A)$ with respect to $\subseteq$.

The previous forcing was considered in [58] for the particular case where $A = \omega$.

Lemma 4.3.2. *Let* $2 \leqslant n \in \omega$ *and* $\mathcal{F}$ *be a 2-capturing construction scheme. Then* $\mathbb{D}_n(\mathcal{F}, A)$ *is ccc for any* $A \subseteq \omega$.

Proof. Let $\mathcal{A}$ be an uncountable subset of $\mathbb{D}_n(\mathcal{F}, A)$. Consider $\mathcal{A}' \subseteq \mathcal{A}$ an uncountable root-tail-tail Δ-system with root R so that any two elements of $\mathcal{A}'$ have the same cardinality. Note that we can enumerate $\mathcal{A}'$ as $\langle p_\alpha \rangle_{\alpha \in \omega_1}$ in such way that $\max(p_\alpha\backslash R) < \min(p_\beta\backslash R)$ whenever $\alpha < \beta$. Since $\mathcal{F}$ is 2-capturing, we can find $\delta < \gamma \in \mathrm{Lim}$ so that the set $\{p_\delta \cup p_{\delta+1}, p_\gamma \cup p_{\gamma+1}\}$ is captured at some level l. The

proof will end by showing the following claim.

Claim: $p = p_\delta \cup p_{\gamma+1} \in \mathbb{D}_n(\mathcal{F}, A)$.

Proof of claim. Let $D \in [p]^n$ which is captured and $h : (p_\delta \cup p_{\delta+1})_{l-1} \longrightarrow (p_\gamma \cup p_{\gamma+1})_{l-1}$ be the increasing bijection. It is easy to see that $h[p_\delta] = p_\gamma$ and $h[p_{\delta+1}] = p_{\gamma+1}$. Therefore,

$$\|\alpha\|_{l-1} = \|h(\alpha)\|_{l-1} < \|\beta\|_{l-1}$$

for each $\alpha \in p_\delta \backslash R$ and $\beta \in p_{\gamma+1} \backslash R$. Thus, $\rho^D < l$ by means of the point (1) of Proposition 2.4.10. In this way, either $D \subseteq p_\delta$ or $D \subseteq p_{\gamma+1}$. As both p_δ and $p_{\gamma+1}$ belong to $\mathbb{D}_n(\mathcal{F}, A)$, it follows that $\rho^D \notin A$. This finishes the proof. $\square$

∎

The following corollary will help us turn $\mathcal{U}_n(\mathcal{F})$ into an ultrafilter.

Corollary 4.3.3. *Let* $2 \leqslant n \in \omega$, $\mathcal{F}$ *be a 2-capturing construction scheme and* $A \subseteq \omega$. *Then there is a condition* $p \in \mathbb{D}_n(\mathcal{F}, A)$ *so that*

$$p \Vdash \text{``} A \notin \mathcal{U}_n(\mathcal{F})^+ \text{''}.$$

Proof. Since $\mathcal{F}$ is 2-capturing, $\mathbb{D}_n(\mathcal{F}, A)$ is a *ccc*-forcing. Furthermore, it is uncountable. Thus, there is $p \in \mathbb{D}_n(\mathcal{F}, A)$ which forces the generic filter to be uncountable. It follows that if G is a generic filter over V so that $p \in G$, then $S_G = \bigcup G$ is an uncountable subset of ω_1 with $\pi_n(S_G) \cap A = \emptyset$. Thus, in $V[G]$, $A \notin \mathcal{U}_{n+1}(\mathcal{F})$.   ∎

Many of the results in the previous chapter already imply that $\mathfrak{m} > \omega_1$ is inconsistent with CA_2. We give a direct proof here for convenience of the reader.

Corollary 4.3.4 (Under $\mathfrak{m} > \omega_1$). *There are no 2-capturing construction schemes for any type.*

Proof. Suppose towards a contradiction that there is a 2-capturing construction scheme, namely $\mathcal{F}$. Then $\mathbb{D}_n(\mathcal{F}, \omega)$ is an uncountable *ccc*-forcing. As $\mathfrak{m} > \omega_1$ such forcing contains an uncountable filter, namely G. Then $S_G = \bigcup G$ is an uncountable subset of ω_1 so that $\pi_2(S_G)$ is empty. This contradicts the fact that $\mathcal{F}$ is 2-capturing, so we are done.

∎

Lemma 4.3.5. *Let* $2 \leqslant n \in \omega$ *and* $\mathcal{F}$ *be an* n*-capturing construction scheme. If* $A \in \mathcal{U}_n(\mathcal{F})^+$ *then* $\mathbb{D}_n(\mathcal{F}, \omega \backslash A)$ n*-preserves* $\mathcal{F}$.

Proof. We will prove this lemma by appealing to the equivalence provided by Lemma 4.0.3. Let $\mathcal{A}$ be an uncountable subset of $\mathbb{D}_n(\mathcal{F}, \omega \backslash A)$ and $\nu : \mathcal{A} \longrightarrow \omega_1$ be an injective function. Given $p \in \mathcal{A}$, let $D_p = p \cup \{\nu(p)\}$ and $\alpha_p = \max(D_p)$. By refining $\mathcal{A}$ if necessary, we may assume that are $j \in \omega$, $a < m_j$, $C \subseteq a+1$ and $b \in C$ so that the following conditions hold for any $p \in \mathcal{A}$:

(a) $\rho^{D_p} \leqslant j$,

(b) $\|\alpha_p\|_j = a$,

(c) $(\alpha_p)_j[C] = p$ and $(\alpha_p)_j(b) = \nu(p)$.

As $A \in \mathcal{U}_n(\mathcal{F})^+$, there are $l \in A$ and $\{p_0, \dots, p_{n-1}\} \in [\mathcal{A}]^n$ for which $\{\, \alpha_{p_i} : i < n \,\}$ is captured at level l. Note that $l > j$ due to the point (b) above. According to Lemma 2.4.11, the family $\{D_{p_i} : i < n\}$ is captured at level l. From this fact and by the point (c), the same holds for both $\{p_0, \dots, p_{n-1}\}$ and $\{\nu(p_0), \dots, \nu(p_{n-1})\}$. The proof follows from the following claim.

<u>Claim</u>: $p = \bigcup_{i<n} p_i \in \mathbb{D}_n(\mathcal{F}, \omega \backslash A)$.

Proof of claim. Let $D \in [p]^n$ which is captured and consider $F \in \mathcal{F}_l$ such that $p \subseteq F$. As $D \subseteq p$, then $\rho^D \leqslant \rho^p = l$. If $\rho^D = l$, we are done because $l \notin \omega \backslash A$. On the other hand, if $\rho^D < l$ then $D \subseteq F_{\Xi_D(l)}$. Since the family $\{p_0, \dots, p_n\}$ is captured at level l, there is $i < n$ so that $F_{\Xi_D(l)} \cap p = p_i$. In this way, $D \subseteq p_i$. Thus, as $p_i \in \mathbb{D}_n(\mathcal{F}, \omega \backslash A)$ then $\rho^D \notin \omega \backslash A$. $\qquad\square$

$\blacksquare$

Lemma 4.3.6. *Let $3 \leqslant n \in \omega$, $\mathcal{F}$ be an n-capturing construction scheme and $A \subseteq \omega$. If $B \in \mathcal{U}_{n-1}(\mathcal{F})^+$, then*

$$\mathbb{D}_n(\mathcal{F}, A) \Vdash \text{ `` } B \in \mathcal{U}_{n-1}(\mathcal{F})^+ \text{ ''}.$$

Proof. Let $\dot{X}$ be a name for an uncountable subset of ω_1 and $q \in \mathbb{D}_n(\mathcal{F}, A)$. Using that $B \in \mathcal{U}_{n-1}(\mathcal{F})^+$ and by arguing in a similar way as in the previous lemmas, we can find a sequence $\langle p_i, \alpha_i \rangle_{i<n-1} \subseteq \mathbb{D}_n(\mathcal{F}, A) \times \omega_1$ so that:

(1) both $\{p_0, \dots, p_{n-2}\}$ and $\{\alpha_0, \dots, \alpha_{n-2}\}$ are captured at some level $l \in B$,

(2) $\alpha_i \neq \alpha_j$ whenever $i \neq j$,

(3) $p_i \leqslant q$ and $p_i \Vdash \text{ `` } \alpha_i \in \dot{X} \text{ ''}$ for each $i < n-1$.

It is easy to see that $p = \bigcup_{i<n-1} p_i$ is a condition in the forcing that we are considering. Furthermore, $p \leqslant q$ and $p \Vdash \text{ `` } \{\alpha_0, \dots, \alpha_{n-1}\} \in [\dot{X}]^{n-1} \text{ ''}$. This means that $p \Vdash \text{ `` } \pi_{n-1}(\dot{X}) \cap B \neq \emptyset \text{ ''}$. Thus, the proof is over. $\qquad\blacksquare$

The following lemma is proved in the exact same way as the previous one. For that reason, we leave the proof to the reader.

Lemma 4.3.7. *Let $\mathcal{F}$ be an n-capturing construction scheme, $B \in \mathcal{U}_n(\mathcal{F})^+$ and $(\langle \mathbb{P}_\xi \rangle_{\xi \leqslant \gamma}, \langle \dot{\mathbb{Q}}_\xi \rangle_{\xi < \gamma})$ be a finite support iteration of ccc forcings so that*

$$\mathbb{P}_\xi \Vdash \text{ `` } \dot{\mathbb{Q}}_\xi \Vdash \text{ `}B \in \mathcal{U}_n(\mathcal{F})^+\text{ '''}.$$

Then $\mathbb{P}_\gamma \Vdash \text{``}B \in \mathcal{U}_n(\mathcal{F})^+ \text{''}$.

By combining all the results we have so far, we get the following theorem.

Theorem 4.3.8. *Let $\mathcal{F}$ be $\mathcal{P}$-n-capturing construction scheme. There is a ccc forcing $\mathbb{P}$ satisfies the following properties:*

$$\mathbb{P} \Vdash \text{``}\mathcal{F} \text{ is n-capturing and } \mathcal{P}\text{-}(n-1)\text{-capturing''},$$

$$\mathbb{P} \Vdash \text{``}\mathcal{U}_n(\mathcal{F}) \text{ is an ultrafilter ''}.$$

In particular, $\mathbb{P}$ forces that $\mathcal{F}$ is not $\mathcal{P}'$-n-capturing for any non-trivial partition $\mathcal{P}'$ of ω.

Proof. Let $\kappa = \mathfrak{c}$. We can construct a finite support iteration $(\langle \mathbb{P}_\xi \rangle_{\xi \leqslant \kappa}, \langle \dot{\mathbb{Q}}_\xi \rangle_{\xi < \kappa})$ of forcings so that given $\xi < \kappa$, the following condition holds:

$$\mathbb{P}_\xi \Vdash \text{``}\dot{\mathbb{Q}}_\xi = \mathbb{D}_n(\mathcal{F}, \dot{A}) \text{ for some } \dot{A} \in [\omega]^\omega \text{ with } \dot{A}, \omega \backslash \dot{A} \in \mathcal{U}_n(\mathcal{F})^{+}\text{''}.$$

By means of Lemma 4.3.2, it follows that $\mathbb{P}_\kappa$ is *ccc*. According to Lemma 4.3.5 and 4.1.7, we have that $\mathbb{P}$ *n*-preserves $\mathcal{F}$. Furthermore, $\mathbb{P} \Vdash \text{`` } \mathcal{P} \subseteq \mathcal{U}_{n-1}(\mathcal{F})^{+}\text{''}$ due to Lemma 4.3.6 and 4.3.7. If we construct the iteration with an appropriate bookkeeping and by making use of Corollary 4.3.3, we can arrange $\mathbb{P}_\kappa$ to force $\mathcal{U}_n(\mathcal{F})$ to be an ultrafilter. This finishes the proof. ∎

Corollary 4.3.9. *Let $1 \leqslant n \in \omega$. It is consistent that there is a construction scheme $\mathcal{F}$ such that:*

- *There is a non-trivial partition of ω, namely $\mathcal{P}$, such that $\mathcal{F}$ is $\mathcal{P}$-n-capturing.*

- *$\mathcal{F}$ is $n+1$-capturing but it is not $\mathcal{P}'$-$n+1$-capturing for any non-trivial partition $\mathcal{P}'$.*

4.4 Ramsey ultrafilters from construction schemes

Let $\mathcal{U}$ be an ultrafilter over ω. Recall that $\mathcal{U}$ is *Ramsey* if for each infinite partition $\mathcal{P} \subseteq \mathcal{U}^*$ of ω, there is $A \in \mathcal{U}$ so that $|A \cap P| \leqslant 1$ for all $P \in \mathcal{P}$. Ramsey ultrafilters, also called selective, play a highly important roll in modern set theory. In [109] Stevo Todorčević presented a clever ways of defining filters over ω_1 by using walks on ordinals. In [115], he used such filters in order to show that the existence of Ramsey ultrafilters follows from $\mathfrak{m} > \omega_1$. This result is quite interesting since the existence of ultrafilters with particular properties usually follows from equalities of the form "*cardinal invariant* $= \mathfrak{c}$" instead of inequalities of the form "*cardinal invariant* $> \omega_1$". In this section we will show that given a 2-capturing construction scheme $\mathcal{F}$, the filter $\mathcal{U}_n(\mathcal{F})$ is a Ramsey ultrafilter under $\mathfrak{m}_\mathcal{F}^n > \omega_1$. Furthermore, we will define the filter $\mathcal{V}(\mathcal{F})$ and show that, under $\mathfrak{m}_\mathcal{F} > \omega_1$, this filter is an ultrafilter over ω_1 which projects into a Ramsey ultrafilter. Beyond the hypothesis imposed over the cardinal invariant $\mathfrak{m}_\mathcal{F}^n$, the most important feature of the following constructions is that both ultrafilters are explicitly definable using a combinatorial structure over ω_1.

Theorem 4.4.1. *Let $2 \leqslant n \in \omega$ and $\mathcal{F}$ be an n-capturing construction scheme for which $\mathfrak{m}_\mathcal{F}^n > \omega_1$. Then $\mathcal{U}_n(\mathcal{F})$ is a Ramsey ultrafilter.*

Proof. Suppose towards a contradiction that $\mathcal{U}_n(\mathcal{F})$ is not an ultrafilter and let $A \subseteq \omega$ be such that $A, \omega \setminus A \in \mathcal{U}_n(\mathcal{F})^+$. According to the Lemmas 4.3.2 and 4.3.5, $\mathbb{D}_n(\mathcal{F}, A)$ is a *ccc* forcing so that $\mathbb{D}_n(\mathcal{F}, A) \Vdash$ " $\mathcal{F}$ is n-capturing ". Hence, its Martin's number is greater than ω_1 due to the hypotheses. In this way, there is an uncountable filter G over $\mathbb{D}_n(\mathcal{F}, A)$. Let $S_G = \bigcup G$. Then S_G is an uncountable subset of ω_1 satisfying that $\pi_n(S_G) \cap A = \emptyset$. This contradicts the fact that $A \in \mathcal{U}_n(\mathcal{F})^+$.

Now, we will show that $\mathcal{U}_n(\mathcal{F})$ is Ramsey. For this purpose, let $\langle P_i \rangle_{i \in \omega}$ be a partition of ω so that $P_i \notin \mathcal{U}_n(\mathcal{F})$ for every $i \in \omega$. It suffices to prove that there is $S \in [\omega_1]^{\omega_1}$ with $|\pi_n(S) \cap P_i| \leqslant 1$ for any $i \in \omega$. Let $\mathbb{P}$ be the forcing consisting of the empty set and all $p \in \text{FIN}(\omega_1)$ so that:
$$|\pi_n(p) \cap A_i| \leqslant 1$$
for each $i \in \omega$. It is easy to see that if G is an uncountable filter over $\mathbb{P}$ then $S_G = \bigcup G$ satisfies the required property. As we are assuming that $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$, the existence of such G will follow if $\mathbb{P}$ is *ccc* and $\mathbb{P} \Vdash$ " $\mathcal{F}$ is n-capturing ". The proof this two facts is similar to ones in Lemmas 4.3.2 and 4.3.5. For that reason, we leave the details to the reader.

$\blacksquare$

Definition 4.4.2 (A square bracket operation). Let $\mathcal{F}$ be a construction scheme. We define $[\![\cdot, \cdot]\!] : [\omega_1]^2 \longrightarrow \omega$ as:
$$[\![\alpha, \beta]\!] = \min((\beta)_{\rho(\alpha, \beta)-1} \setminus \alpha)$$
for each $\alpha < \beta$. Given $S \subseteq \omega_1$, we also define
$$(\!|S|\!) = \{ [\![\alpha, \beta]\!] : \alpha, \beta \in S \text{ and } \{\alpha, \beta\} \text{ is captured} \}$$

Remark 4.4.3. Suppose that $\alpha < \beta \in \omega_1$ and let $l = \rho(\alpha, \beta)$. If $F \in \mathcal{F}_l$ is such that $\{\alpha, \beta\} \subseteq F$, then

$$[\![\alpha, \beta]\!] = \min(F_{\Xi_\beta(l)} \setminus R(F)).$$
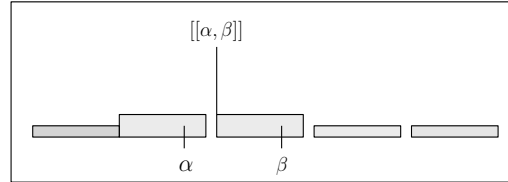
Remark 4.4.4. As a consequence of the previous remark we have that if $D_0, D_1 \in \text{FIN}(\omega_1)$ are disjoint sets for which $\{D_0, D_1\}$ is captured, then
$$[\![D_0(i), D_1(i)]\!] = [\![D_0(j), D_1(j)]\!]$$
for all $i, j < |D_0|$.

Definition 4.4.5 (The square-bracket filter). Let $\mathcal{F}$ be a construction scheme. We define $\mathcal{V}(\mathcal{F})$ as the set of all $A \subseteq \omega_1$ for which there is $S \in [\omega_1]^{\omega_1}$ such that
$$(\!|S|\!) \subseteq A.$$

Recall that an filter over ω_1 is uniform whenever all of its elements are uncountable.

Lemma 4.4.6. *Let $\mathcal{F}$ be a 2-capturing construction scheme. Then $\mathcal{V}(\mathcal{F})$ is a uniform filter.*

Proof. Since $\mathcal{F}$ is 2-capturing, each element of $\mathcal{V}(\mathcal{F})$ is non-empty. In order to show that $\mathcal{V}(\mathcal{F})$ is a filter, it is enough to show that the family $\{(S) : S \in [\omega_1]^{\omega_1}\}$ is downwards directed. Let $S, S' \in [\omega_1]^{\omega_1}$. We will prove that there is $A \in [\omega_1]^{\omega_1}$ so that $(A) \subseteq (S) \cap (S')$. Just as in the Lemma 4.2.5, we can consider a sequence $\langle \alpha_\xi, \beta_\xi \rangle_{\xi \in \omega_1} \subseteq S \times S'$ so that $\alpha_\xi < \beta_\xi < \alpha_{\xi+1}$ for each $\xi \in \omega_1$. We may assume without loss of generality that there is $k \in \omega$ so that the following properties hold for any two distinct $\xi, \mu \in \omega_1$:

(1) $\rho(\alpha_\xi, \beta_\xi) < k$,

(2) $\|\alpha_\xi\|_k = \|\alpha_\mu\|_k$ and $\|\beta_\xi\|_k = \|\beta_\mu\|_k$. In particular, this means that $\rho(\beta_\xi, \beta_\mu) > k$.

We claim that $A = \{\beta_\xi : \xi \in \omega_1\}$ works. Trivially $(A) \subseteq (S')$ because $A \subseteq S'$. In order to prove that $(A) \subseteq (S)$, let $\delta \in (A)$. Then there are $\xi < \mu \in \omega_1$ for which $\{\beta_\xi, \beta_\mu\}$ is captured at some level l and $[\![\beta_\xi, \beta_\mu]\!] = \delta$. The properties (1) and (2) stated above imply that the family $\{\{\alpha_\xi, \beta_\xi\}, \{\alpha_\mu, \beta_\mu\}\}$ is also captured at level l (see Lemma 2.4.11). In particular, $\{\alpha_\xi, \alpha_\mu\}$ is captured. Furthermore, $\delta = [\![\beta_\xi, \beta_\mu]\!] = [\![\alpha_\xi, \alpha_\mu]\!]$ by means of the Remark 4.4.4. As $\alpha_\xi, \alpha_\mu \in S$, we have shown that $\delta \in (S)$. Thus, the proof is over. ∎

Theorem 4.4.7. *Let $\mathcal{F}$ be a 2 capturing construction scheme for which $\mathfrak{m}_\mathcal{F} > \omega_1$. Then $\mathcal{V}(\mathcal{F})$ is an ultrafilter.*

Proof. Let $A \in \mathcal{V}(\mathcal{F})^+$. We will show that $A \in \mathcal{V}(\mathcal{F})$. For this purpose, let us consider the forcing

$$\mathbb{P} = \{ p \in [\omega_1]^{<\omega} : (p) \subseteq A \}$$

ordered by reverse inclusion. It is straightforward that if $G \subseteq \mathbb{P}$ is an uncountable filter, then $S_G = \bigcup G$ is an uncountable subset of ω_1 for which $(S_G) \subseteq A$. In order to guarantee the existence of such filter, it is enough to prove that $\mathbb{P}$ is *ccc* and 2-preserves $\mathcal{F}$. This is because we are assuming that $\mathfrak{m}_\mathcal{F} > \omega_1$.

<u>Claim</u>: $\mathbb{P}$ is *ccc* and 2-preserves $\mathcal{F}$.

Proof of claim. Let $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$ and $\nu : \mathcal{A} \longrightarrow \omega_1$ bi an injective function. For each $p \in \mathcal{A}$ put $D_p = p \cup \{\nu(p)\} \alpha_p = \max(D_p)$. By refining $\mathcal{A}$ if necessary, we may assume that there are $j \in \omega$, $a < m_j$, $C \subseteq a + 1$ and $b \in C$ so that the following conditions hold for any $p \in \mathcal{A}$:

(a) $\rho^{D_p} \leqslant j$,

(b) $\|\alpha_p\|_j = a$,

(c) $(\alpha_p)_j[C] = p$ and $(\alpha_p)_j(b) = \nu(p)$.

Since $A \in \mathcal{V}(\mathcal{F})^+$, there are $p_0, p_1 \in \mathcal{A}$ for which $\{\alpha_{p_0}, \alpha_{p_1}\}$ is captured at some level l and $[\![\alpha_{p_0}, \alpha_{p_1}]\!] \in A$. According to Lemma 2.4.11, $\{D_{p_0}, D_{p_1}\}$ is also captured at level l. From this fact and by point (c), the same holds for $\{p_0, p_1\}$ and $\{\nu(p_0), \nu(p_1)\}$. In order to finish the claim, we will show that $q = p_0 \cup p_1 \in \mathbb{P}$. Indeed, let $\alpha < \gamma \in q$ be such that $\{\alpha, \gamma\}$ is captured. If $\rho(\alpha, \gamma) < l$, then there is $i \in 2$ for which $\alpha, \gamma \in p_i$.

Thus, $[\![\alpha, \gamma]\!] \in A$. On the other hand, if $\rho(\alpha, \gamma) = l$, let $F \in \mathcal{F}_l$ be such that $q \subseteq F$. By Remark 4.4.3, we have that

$$[\![\alpha, \gamma]\!] = F_1 \backslash R(F) = [\![\alpha_{p_0}, \alpha_{p_1}]\!] \in A.$$

$\square$

$\blacksquare$

Suppose that $\mathcal{F}$ is as in the previous Theorem. Since ω_1 is not measurable, there is a surjective function $\pi : \omega_1 \longrightarrow \omega$ so that $\pi^{-1}[\{n\}] \notin \mathcal{V}(\mathcal{F})$ for each $n \in \omega$. From this, it follows that the family

$$\pi\mathcal{V}(\mathcal{F}) = \{A \in \omega \ : \ \pi^{-1}[A] \in \mathcal{V}(\mathcal{F})\}$$

is a non-principal ultrafilter over ω. We end this chapter by proving that it is a Ramsey ultrafilter as well.

Theorem 4.4.8. *Let $\mathcal{F}$ be a 2-capturing construction scheme for which $\mathfrak{m}_\mathcal{F} > \omega_1$. then $\pi\mathcal{V}(\mathcal{F})$ is a Ramsey ultrafilter.*

Proof. Let $\langle P_i \rangle_{i \in \omega}$ be a partition of ω so that $P_i \notin \pi\mathcal{V}(\mathcal{F})$ for each $\in \omega$. We define the forcing $\mathbb{P}$ as the set of all $p \in [\omega_1]^\omega$ so that

$$|\pi[(\!|p|\!)] \cap P_i| \leqslant 1$$

for each $i \in \omega$. We order $\mathbb{P}$ with the reverse inclusion. Note that if $G \subseteq \mathbb{P}$ is an uncountable filter, then S_G is an uncountable subset of ω_1 such that $|\pi[(\!|S_G|\!)] \cap P_i| \leqslant 1$ for each i. The existence of such filter follows from the following claim.

<u>Claim</u>: $\mathbb{P}$ is *ccc* and 2-preserves $\mathcal{F}$.

Proof of claim. Let $\mathcal{A}$ be an uncountable subset of $\mathbb{P}$ and $\nu : \mathcal{A} \longrightarrow \omega_1$ be an injective function. Given $p \in \mathcal{A}$, let $D_p = p \cup \{\nu(p)\}$ and $\alpha_p = \max(p)$. By refining $\mathcal{A}$ if necessary, we may assume that there are $j \in \omega$, $a < m_j$, $C \subseteq a + 1$ and $b \in C$ so that the following conditions hold for any $p \in \mathcal{A}$:

(a) $\rho^{D_p} \leqslant j$,

(b) $\|\alpha_p\|_j = a$,

(c) $(\alpha_p)_j[C] = p$ and $(\alpha_p)_j(b) = \nu(p)$.

Furthermore, we may suppose that there is $E \subseteq \omega$ so that $\pi[(\!|p|\!)] = E$ for each $p \in \mathcal{A}$. Let $I = \{i \in \omega \ : \ E \cap P_i \neq \emptyset\}$. Note that I is finite. Therefore,

$$A = \omega \backslash (\bigcup_{i \in I} P_i) \in \pi\mathcal{V}(\mathcal{F}).$$

In particular, $A \cap \pi[(\!|\alpha_p \ : \ p \in \mathcal{A}|\!)] \neq \emptyset$. Thus, there are $p_0, p_1 \in \mathcal{A}$ so that $\{\alpha_{p_0}, \alpha_{p_1}\}$ is captured at some level l and $\pi([\![\alpha_{p_0}\alpha_{p_1}]\!]) \in A$. By arguing in the same way as in the

previous theorem, we can conclude that both $\{\nu(p_0), \nu(p_1)\}$ and $\{p_0, p_1\}$ are captured. Even more, for $q = p_0 \cup p_1$ we have that

$$(\!|q|\!) = \{[\![\alpha_{p_0}, \alpha_{p_1}]\!]\} \cup (\!|p_0|\!) \cup (\!|p_1|\!).$$

In virtue of this, $\pi[(\!|q|\!)] = \{\pi([\![\alpha_{p_0}, \alpha_{p_1}]\!])\} \cup E$. As $\pi[\![\alpha_{p_0}, \alpha_{p_1}]\!] \notin A$, it easily follows that $q \in \mathbb{P}$. This finishes the proof. $\qquad\square$

$\blacksquare$

Chapter 5

On the existence of construction and capturing schemes

The main goal of this chapter is to present a complete proof that the capturing axiom $FCA(part)$ follows from Jensen's $\diamondsuit$-principle. In order to do that, we will first analyse in more depth some structural properties regarding construction schemes. Among other things, we will conclude that if the type is good, any construction scheme over ω_1 can be reconstructed from an $\subseteq$-increasing chain of construction schemes over the countable limit ordinal and viseversa. That is, any increasing chain of construction scheme over the countable limit ordinals determines a construction scheme over ω_1. Furthermore, the first element of such chain is uniquely determined. In other words, there is a unique construction scheme over ω (which we call $\mathcal{F}(\omega)$). In Section 5.1, we will define the forcing $\mathbb{P}(\mathcal{F})$. We will use this forcing as a tool for extending a construction scheme from a countable limit ordinal to the next one. In Subsection 5.1.1, we will use the forcing $\mathbb{P}(\mathcal{F})$ to show that construction schemes (over ω_1) of any given good type exist without the assumption of any extra axioms. In Section 5.2, we will prove that the $\diamondsuit$-principle implies $FCA(part)$. Finally, in Section 5.3, we will prove that PID is incompatible with the existence of a 2-capturing construction scheme.

For the rest of this chapter, we fix a type $\tau = \langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega}$.

Definition 5.0.1 (The restriction of a scheme). Let $\mathcal{F}$ be a construction scheme over X. For any $Y \subseteq X$, we define the *restriction of $\mathcal{F}$ to Y* as

$$\mathcal{F}|_Y = \{F \in \mathcal{F} \,:\, F \subseteq Y\}.$$

Remark 5.0.2. If $Y \in \mathcal{F}$, then $\mathcal{F}|_Y$ is a construction scheme over Y.

Remark 5.0.3. If $Y \subseteq X$ and $k \in \omega$, then $(\mathcal{F}|_Y)_k = \{F \in \mathcal{F}_k \,|\, F \subseteq Y\}$.

Remark 5.0.4. If $\mathcal{F}$ is a construction scheme over a finite set X, then $X \in \mathcal{F}$. This is due to the point (a) of Definition 2.2.2.

Remark 5.0.5. If $\mathcal{F}$ is a construction scheme over X, and $Y \subseteq X$ is such that $\mathcal{F}|_Y$ is a construction scheme over Y, then $\rho_{\mathcal{F}}|_{Y^2} = \rho_{\mathcal{F}|_Y}$.

In the two following propositions we characterize the finite sets of ordinals which admit construction schemes.

Proposition 5.0.6. *If X is a finite set of ordinals, then there is at most one construction scheme over X of type τ.*

Proof. The proof is carried by induction over $m = |X|$. If $m = 1$, the result is immediate. For the inductive step, suppose that we have proved the proposition for each set of ordinals of cardinality at most m and consider X such that $|X| = m+1$. Let $\mathcal{F}$ and $\mathcal{F}'$ be construction schemes over X. According to the Remark 5.0.4, $X \in \mathcal{F} \cap \mathcal{F}'$. Thus, there are $k, k' \in \omega$ so that $X \in \mathcal{F}_k$ and $X \in \mathcal{F}'_{k'}$. By the point (b) of Definition 2.2.2, $m_k = |X| = m_{k'}$. In this way, $k = k'$. Furthermore, since $|X| = m + 1 > 1$ then $k > 0$. Therefore, due to the point (d) of Definition 2.2.2, we can take $\langle X_i \rangle_{i<n_k}$ and $\langle X'_i \rangle_{i<n_k}$ to be the canonical decompositions of X inside $\mathcal{F}$ and $\mathcal{F}'$ respectively. In the Remark 2.2.3, we explicitly described the canonical decomposition of a set in terms of the type. From this description it is easy to see that $X_i = X'_i$ for any $i < n_k$. By means of Proposition 2.1.20 we can deduce that

$$\mathcal{F} = \{X\} \cup \big(\bigcup_{i<n_k} \mathcal{F}|_{X_i} \big),$$

$$\mathcal{F}' = \{X\} \cup \big(\bigcup_{i<n_k} \mathcal{F}'|_{X_i} \big).$$

But $\mathcal{F}|_{X_i} = \mathcal{F}'|_{X_i}$ for each i due to the inductive hypotheses. Consequently, $\mathcal{F} = \mathcal{F}'$. ∎

Proposition 5.0.7. *Let $k \in \omega$. If X is a finite set of ordinals and $|X| = m_k$ then there is a unique construction scheme over X of type τ.*

Proof. The uniqueness part was showed in Proposition 5.0.6. The "existence" part of proof will be carried by induction over k.

Base step: If $k = 0$, then $m_k = 1$. It should be clear that $\mathcal{F} = \{X\}$ is the only construction scheme over X.

Inductive Step: Suppose that the proposition is true for some $k \in \omega$ and each set of ordinals of size m_k. Let X be a set of ordinals with $|X| = m_{k+1}$. Given $i < n_{k+1}$, let us define $a_i = r_{k+1} + i \cdot (m_k - r_{k+1})$ and

$$F_i = X[r_{k+1}] \cup X[\,[a_i,\, a_{i+1})\,].$$

Note that each F_i has cardinality m_k. Therefore, by the inductive hypotheses there is a unique construction scheme $\mathcal{F}_i$ over F^i of type τ. Let

$$\mathcal{F} = \{X\} \cup \Big(\bigcup_{i<n_{k+1}} \mathcal{F}^i \Big).$$

We claim that $\mathcal{F}$ is the construction scheme over X that we are looking for. $\mathcal{F}$ satisfies condition (a) of Definition 2.2.2 because $X \in \mathcal{F}$. It satisfies condition (b) of such definition because for each $l \in \omega$,

$$
\mathcal{F}_l = \begin{cases} \emptyset & \text{if } l > k+1 \\ \{X\} & \text{if } l = k+1 \\ \bigcup_{i<n_{k+1}} \mathcal{F}_l^i & \text{if } l \leqslant k \end{cases}
$$

Let us fix $l \in \omega$. We proceed to prove that the conditions (c) and (d) of Definition 2.2.2 applied to l also hold. Note that when $l > k+1$ there is nothing to verify. This is because, in this case, $\mathcal{F}_l$ is empty. Therefore, we will assume that $l \leqslant k+1$.

Proof of (c). If $l = k+1$, condition (c) trivially holds for l because $\mathcal{F}_l = \{X\}$. Therefore, we can assume that $l \leqslant k$. In this case, let $F, E \in \mathcal{F}_l$. We need to show that $E \cap F \sqsubseteq F$. According to the formula for $\mathcal{F}_l$ written above, we know that there are $i, j < n_{k+1}$ for which $E \in \mathcal{F}_l^i$ and $F \in \mathcal{F}_l^j$. If $i = j$, we are done since $\mathcal{F}^i$ is already a construction scheme. So suppose that $i \neq j$ and consider $\phi : F_i \longrightarrow F_j$ the increasing bijection. It is not hard to see that $\mathcal{F}^\phi = \{\phi[G] : G \in \mathcal{F}^i\}$ is a construction scheme over F_j of type τ. Therefore, $\mathcal{F}^j = \mathcal{F}^\phi$ by virtue of the inductive hypotheses. In particular, there is $G \in \mathcal{F}_l^i$ for which $F = \phi[G]$. According to the definitions of F_i and F_j it follows that $F \cap X[r_{k+1}] = G \cap X[r_{k+1}]$ and $E \cap F \subseteq X[r_{k+1}]$. In this way,

$$
E \cap F = E \cap F \cap X[R_{k+1}] = E \cap G \cap X[r_{k+1}].
$$

Note that $E \cap G \sqsubseteq G$ because $E, G \in \mathcal{F}_l^i$. Furthermore, since $X[r_{k+1}] \sqsubseteq F_j$ and $F \subseteq F_j$, then $F \cap X[r_{k+1}] \sqsubseteq F$. Therefore,

$$
E \cap F = E \cap G \cap X[r_{k+1}] \sqsubseteq G \cap X[r_{k+1}] = F \cap X[r_{k+1}] \sqsubseteq F.
$$

$\square$

Proof of (d). Let $F \in \mathcal{F}_l$. If $l = k+1$ then $F = X$. In this case, it is easy to see that the sequence $\langle F_i \rangle_{i<n_{k+1}}$ satisfies all the required properties. On the other hand, if $l < k+1$ then $F \in \mathcal{F}^i$ for some $i < n_{k+1}$. Therefore, the existence of the canonical decomposition of F is assured because $\mathcal{F}^i$ is already a construction scheme. $\square$

$\blacksquare$

Definition 5.0.8 (Unique scheme (finite version))**.** If $k \in \omega$ and X is a finite set of ordinals of size m_k, we call $\mathcal{F}(X)$ the unique construction scheme over X of type τ.

Remark 5.0.9. If $\mathcal{F}$ is a construction scheme over X and $F \in \mathcal{F}$, then $\mathcal{F}|_F = \mathcal{F}(F)$.

Remark 5.0.10. If X and Y are finite sets of cardinality m_k and $\phi : X \longrightarrow Y$ is the increasing bijection, then $\mathcal{F}(Y) = \{\phi[F] : F \in \mathcal{F}(X)\}$.

Now, we will show that ω also admits a unique construction scheme (of type τ). The uniqueness of the scheme will be proved using the following result.

Lemma 5.0.11. *Suppose that X and Y are two sets of ordinals for which $X \subseteq Y$. Let $\mathcal{F}$ and $\mathcal{G}$ be construction schemes over X and Y respectively. If $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}|_X$.*

Proof. By definition we have that $\mathcal{F} \subseteq \mathcal{G}|_X$. We proceed to prove the remaining inclusion. For this, let $G \in \mathcal{G}|_X$, $k = \rho_{\mathcal{G}}^G$ and $\gamma = \max(G)$. By virtue of Proposition 2.2.9, there is $F \in \mathcal{F}_k$ such that $\gamma \in F$. By the hypotheses, $F \in \mathcal{G}$. Furthermore, $F \in \mathcal{G}_k$ because $|F| = m_k$. In this way, $F \cap G \sqsubseteq G$ by means of the property (c) of Definition 2.2.2. Since $\gamma \in F \cap G$, this means that $F \cap G = G$. Finally, since $|F| = m_k = |G|$, we conclude that $F = G$. In other words, $G \in \mathcal{F}$. ∎

Remark 5.0.12. If X, Y, $\mathcal{F}$ and $\mathcal{G}$ are as in the previous lemma, then $\rho_{\mathcal{F}} = \rho_{\mathcal{G}}|_{X^2}$.

Lemma 5.0.13. *Let $\mathcal{B}$ be a family of sets of ordinals which is totally ordered with respect to $\subseteq$. Suppose that $\langle \mathcal{F}^X \rangle_{X \in \mathcal{B}}$ is a sequence so that for any $X, Y \in \mathcal{B}$ with $X \subseteq Y$, the following conditions hold:*

- *$\mathcal{F}^X$ is a construction scheme over X,*

- *$\mathcal{F}^Y|_X = \mathcal{F}^X$ (or equivalently, $\mathcal{F}^X \subseteq \mathcal{F}^Y$).*

Then $\mathcal{F} = \bigcup_{X \in \mathcal{B}} \mathcal{F}^X$ is a construction scheme over $Z = \bigcup \mathcal{B}$.

Proof. $\mathcal{F}$ satisfies the point (a) of Definition 2.2.2 because $\mathcal{B}$ is totally ordered. In order to finish, note that $\mathcal{F}_k = \bigcup_{X \in \mathcal{B}} \mathcal{F}_k^X$ for any $k \in \omega$. The remaining points of Definition 2.2.2 hold for $\mathcal{F}$ as a consequence of this observation. ∎

Lemma 5.0.14. *Let X be a set of ordinals, $\mathcal{F}$ be a construction scheme over X and $k < l \in \omega$. If $F \in \mathcal{F}_l$ and $\alpha \in F$, then there is (a unique) $E \in \mathcal{F}_k$ such that:*

(a) *$E \subseteq F$,*

(b) *$\alpha \in E$,*

(c) *$E \backslash \alpha$ is an interval in F.*

Proof. The uniqueness of E holds because $E \cap \alpha = (\alpha)_k^-$ and $E \backslash \alpha$ is the unique interval of F of size $m_k - \|\alpha\|_k$ starting at α. The proof of the existence carried by induction over l starting at $k + 1$.

Base step: Suppose that $l = k + 1$. In the case where $\alpha \in R(F)$ let $E = F_0$. Otherwise, let $E = F_{\Xi_\alpha(l)}$. It is straightforward that E satisfies all the required conditions in both cases.

Inductive step: Suppose that we have proved the lemma for some $l > k$. We will now prove it for $l + 1$. So let $F \in \mathcal{F}_{l+1}$ be such that $\alpha \in F$. First we deal with the case where $\alpha \in R(F)$. Note that $\alpha \in F_0$, so according to the inductive hypotheses, there is $E \in \mathcal{F}_k$ such that $E \subseteq F_0$, $\alpha \in E$ and $E \backslash \alpha$ is an interval in F_0. Since F_0 is already an interval in F, then $E \backslash \alpha$ is also an interval. Now we deal we the case where $\alpha \notin R(F)$.

In here, let $i = \Xi_\alpha(l)$. Then $\alpha \in F_i \backslash R(F)$. Again, by the inductive hypotheses, there is $E \in \mathcal{F}_k$ such that $E \subseteq F_i$, $\alpha \in E$ and $E \backslash \alpha$ is an interval in F_i. Observe that $F_i \backslash R(F)$ is an interval in F and $E \backslash \alpha$ is contained in it. Therefore, $E \backslash \alpha$ must be an interval in F. This finishes the proof. ∎

Corollary 5.0.15. *Let $\mathcal{F}$ be a construction scheme over X. Suppose that $\alpha \in X$ and $k \in \omega$. Then there is (a unique) $E \in \mathcal{F}_k$ such that $\alpha \in E$ and $E \backslash \alpha$ is an interval in X.*

Proof. Let $m = m_k - \|\alpha\|_k$. Note that $|\,[\alpha, \infty)_X| \geqslant m$. This is because there is at least one $F' \in \mathcal{F}$ with $\alpha \in F'$. For such F' we have that $\alpha \cap F' = (\alpha)_k^-$. Hence, $m = |F \backslash \alpha| \leqslant |X|$.

Since X is a well-ordered set and $|[\alpha, \infty)_X| \geqslant m$, we can take P the unique interval in $[\alpha, \infty)_X$ of size m with $\alpha \in P$. Let $l = \max(k + 1, \rho^P)$. Then there is $F \in \mathcal{F}_l$ such that $P \subseteq F$. According to Lemma 5.0.14, there is $E \in \mathcal{F}_k$ such that $E \subseteq F$, $\alpha \in F$ and $E \backslash \alpha$ is an interval in F. Note that both $E \backslash \alpha$ and P are intervals in F of size m having α as their minimum. In this way, $E \backslash \alpha = P$. Therefore, $E \backslash \alpha$ is an interval in X. This finishes the proof. ∎

Remark 5.0.16. If we apply the previous corollary to the case where X is an ordinal, say γ, we get the following: For any $\alpha \in \gamma$ and $k \in \omega$, we have that

$$(\alpha)_k^- \cup [\alpha, \alpha + (m_k - \|\alpha\|_k))$$

is an element of $\mathcal{F}_k$.

Theorem 5.0.17. *There is a unique construction scheme over ω of type τ. Furthermore, $\{m_k : k \in \omega\}$ is contained in such scheme.*

Proof. Proof of existence. For any $k \in \omega$, let $\mathcal{F}(m_k)$ be the only construction scheme over m_k. Note that m_k is the first element of the canonical decomposition of m_{k+1} inside $\mathcal{F}(m_{k+1})$. This means that $m_k \in \mathcal{F}(m_{k+1})$. Therefore, $\mathcal{F}(m_k) = \mathcal{F}(m_{k+1})|_{m_k}$. In this way, by Lemma 5.0.13, the family $\mathcal{F} = \bigcup_{k \in \omega} \mathcal{F}(m_k)$ is a construction scheme over $\omega = \bigcup_{k \in \omega} m_k$. □

Proof of uniqueness. Suppose that $\mathcal{F}'$ is another construction scheme over ω. Let $k \in \omega$. According to the Remark 5.0.16, the set

$$(0)_k^- \cup [0, 0 - (m_k - \|0\|_k)) = m_k$$

Is an element of $\mathcal{F}'$. Thus, $\mathcal{F}(m_k) = \mathcal{F}'|_{m_k} \subseteq \mathcal{F}'$. Since this is true for any k, we have that $\mathcal{F} \subseteq \mathcal{F}'$. But then $\mathcal{F} = \mathcal{F}'$ due to Lemma 5.0.11. □

∎

Definition 5.0.18 (Unique scheme over ω)**.** We will call $\mathcal{F}(\omega)$ the only construction scheme over ω of type τ.

In order to build construction schemes over ω_1, we will use the same technique as when building $\mathcal{F}(\omega)$. That is, we will define such schemes by taking unions of schemes over distinct sets of countable sets of ordinals. In the next two propositions, we show that there is essentially one way of choosing those countable sets.

Proposition 5.0.19. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ of type τ. If $\delta \leqslant \gamma$ is a limit ordinal, then $\mathcal{F}|_\delta$ is a construction scheme over δ of type τ. In particular, $\mathcal{F}(\omega) \subseteq \mathcal{F}$ whenever γ is infinite.*

Proof. It is straightforward that $\mathcal{F}|_\delta$ satisfies conditions (b), (c) and (d) of Definition 2.2.2. In order to prove that condition (a) also holds, take an arbitrary $A \in \mathrm{FIN}(X)$ and let $\alpha = \max(A)$. We need to prove that there is $F \in \mathcal{F}|_\delta$ such that $A \subseteq F$. Indeed, according to the Remark 5.0.16, we know that

$$F = (\alpha)_k^- \cup [\alpha, \alpha + (m_k - \|\alpha\|_k))$$

is an element of $\mathcal{F}$ for $k = \rho^A$. Since δ is a limit ordinal it follows that $F \subseteq \delta$ (that is, $F \in \mathcal{F}|_\delta$). Furthermore, $A \subseteq (\alpha)_k$ so $A \subseteq F$. ∎

Proposition 5.0.20. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ of type τ. If $X \subseteq \gamma$ is an infinite subset of γ and $\mathcal{F}|_X$ is a construction scheme over X, then X is an ordinal. Furthermore, if τ is a good type then X is limit.*

Proof. In order to show that X is an ordinal, it is enough to prove that X is an initial segment of γ. For this, let $\alpha < \beta \in \gamma$ be such that $\beta \in X$. Consider $k = \rho(\alpha, \beta)$. Since X is infinite, there is $F \in (\mathcal{F}|_X)_k$ such that $\beta \in F$. This is due to Proposition 2.2.9. According to the Remark 5.0.3, $F \in \mathcal{F}_k$. Therefore, $\alpha \in F$. In particular $\alpha \in X$, so we are done.

Now, suppose that τ is a good type. We will prove that X is limit. Assume towards a contradiction that this is not the case. Let $\alpha = \max(X)$ and fix $k > \rho(0, \alpha)$ for which $r_k = 0$. Observe that if $F \in \mathcal{F}_k$ is such that $\alpha \in F$, then $\alpha = \max(F)$. In particular, $\alpha \in F_{n_k-1}$. Furthermore, since $k > \rho(0, \alpha)$ then 0 also belongs to F_{n_k-1}. Lastly, $F_0 < F_{n_k-1}$ because $r_k = 0$. This implies that every element of F_0 is smaller than 0, which is a contradiction. Thus, the proof is over. ∎

In summary, by means of the two previous propositions we know that if τ is a good type and $\mathcal{F}$ is a construction scheme over and ordinal γ then γ is either a m_k for some $k \in \omega$, or γ is limit. Furthermore, the set of all $X \in \mathscr{P}(\gamma)$ for which $\mathcal{F}|_X$ is a construction scheme can be fully described as

$$\mathcal{F} \cup (\mathrm{Lim} \cap (\gamma + 1)).$$

In the case where γ is infinite, the previous family has ω as a special member. This is because, according to the Theorem 5.0.17, $\mathcal{F}|_\omega$ is simply $\mathcal{F}(\omega)$. This scheme in ω stands out not only because of its uniqueness but because each m_k belongs to it. Therefore, m_k not only acts as the common cardinality of all elements in $\mathcal{F}_k$, but it is the "most canonical" member of such family. Actually, this line of thought transfers to all natural numbers. Concretely, since the initial segments of closed sets are closed according to Corollary 2.1.8, we have the following corollary.

Corollary 5.0.21. *Let $\mathcal{F}$ be a construction scheme over an infinite ordinal γ. Then $n \in \omega$ is a closed set with respect to ρ. Furthermore, $n \in (m_{\rho^n - 1}, m_{\rho^n}]$ whenever $n > 0$.*

The metric structure of closed sets is fully determined by their cardinality (see Lemma 2.1.27 and Theorem 2.1.26). In particular, for each $\beta \in \gamma$ and $l \in \omega$, the set $(\beta)_l$ is ρ-isomorphic to $\|\beta\|_l + 1 = [0, \|\beta\|_l]$ (recall $\|\beta\|_l = |(\beta)_l^-|$). It will be convenient to name and explicitly describe the increasing bijections between these two sets.

Definition 5.0.22. Let $\mathcal{F}$ be a construction scheme over a limit ordinal γ. Given $\beta \in \gamma$ and $l \in \omega$, we define $\phi_l^\beta : (\beta)_l \longrightarrow \|\beta\|_l + 1$ as:

$$\phi_l^\beta(\alpha) = \|\alpha\|_l.$$

The inverse function of ϕ_l^β is given by:

$$(\phi_l^\beta)^{-1}(i) = (\beta)_l(i).$$

Proposition 5.0.23. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ. Given $\alpha, \beta \in \gamma$ and $k \leqslant l \in \omega$, the following conditions hold:*

(a) $\|\beta\|_k = \| \|\beta\|_l \|_k$,

(b) *If $l \geqslant \rho(\alpha, \beta)$, then $\rho(\alpha, \beta) = \rho(\|\alpha\|_l, \|\beta\|_l)$*

(c) *If $l \geqslant \Delta(\alpha, \beta)$, then $\Delta(\alpha, \beta) = \Delta(\|\alpha\|_l, \|\beta\|_l)$.*

We can rewrite the point (a) of the previous proposition in a more useful way.

Lemma 5.0.24. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ. Given $\beta \in \gamma$ and $k \leqslant l \in \omega$, we have that*

$$\phi_k^{\|\beta\|_l} \circ \phi_l^\beta = \phi_k^\beta.$$

5.1 The forcing $\mathbb{P}(\mathcal{F})$ and schemes in ZFC

In this section we will present an incarnation of the Cohen forcing which was first considered in [112]. This forcing will serve the purpose of extending a construction scheme from a countable limit ordinal to the next one, and will play a key roll in the arguments of the next sections of this chapter. As an application, we will show in subsection 5.1.1 that there is a construction scheme over ω_1 of any given type. All of this section is based in [112].

For the rest of this section, let us fix a construction scheme $\mathcal{F}$ of type τ over a limit ordinal γ.

Definition 5.1.1 (The forcing $\mathbb{P}(\mathcal{F})$). We define $\mathbb{P}(\mathcal{F})$ as the forcing consisting of the empty set and of all $p \in \text{FIN}(\gamma + \omega)$ with the following properties:
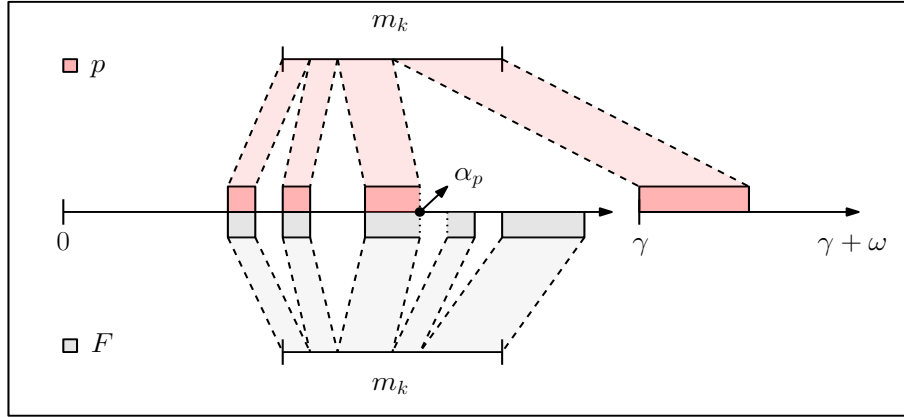
(I) There is $k_p \in \omega$ such that $|p| = m_{k_p}$.

(II) There is $F \in \mathcal{F}_{k_p}$ such that $p \cap \gamma \sqsubseteq F$.

(III) $p \cap [\gamma, \gamma + \omega)$ is an initial segment of $[\gamma, \gamma + \omega)$.

Whenever $p \cap \gamma \neq \emptyset$ (even if p is not a condition of $\mathbb{P}(\mathcal{F})$), we let $\alpha_p = \max(p \cap \gamma)$. Additionally, for each $k \in \omega$ we let $\mathbb{P}_k(\mathcal{F}) = \{\, p \in \mathbb{P}(\mathcal{F}) \, : \, k_p = k \,\}$. The order on $\mathbb{P}(\mathcal{F})$ is given by

$$p \leqslant q \text{ if and only if } q \in \mathcal{F}(p) \text{ or } q = \emptyset.$$

Note that $\mathbb{P}(\mathcal{F})$ is always countable. Therefore, it is forcing equivalent to the Cohen forcing.

In here, p represents an element of $\mathbb{P}_k(\mathcal{F})$ and F represents an element of $\mathcal{F}_k$ which testifies the condition (II) of Definition 5.1.1 for p.

Remark 5.1.2. From Lemma 2.2.13 it follows that the condition (II) of Definition 5.1.1 is equivalent to:

$$(\text{II})^* \ p \cap \gamma = (\alpha_p)_{k_p},$$

in the case where $p \cap \gamma \neq \emptyset$.

In general, if $p \in \mathbb{P}(\mathcal{F})$, there are many elements $F \in \mathcal{F}_{k_p}$ testifying the condition (II) of Definition 5.1.1 with respect to p. However, it turns out that there is a canonical one. Such F will be useful in many of the arguments involving the forcing $\mathbb{P}(\mathcal{F})$. In the following definition, we explicitly describe it.

Definition 5.1.3 (The reduction operation)**.** If $p \in \text{FIN}(\gamma + \omega)$ and $\delta \leqslant \gamma$, we define the *reduction of p to δ* as follows:

$$red_\delta(p) = \begin{cases} (p \cap \delta) \cup [\max(p \cap \delta) + 1, \max(p \cap \delta) + |p \backslash \delta| + 1) & \text{if } p \cap \delta \neq \emptyset \\ |p| & \text{if } p \cap \delta = \emptyset \end{cases}$$

Remark 5.1.4. The reduction operation is closely related to the Corollary 5.0.15. Suppose that we take $\alpha \in \gamma$ and $k \in \omega$. According to the corollary, we know that there is a unique $E \in \mathcal{F}_k$ so that $\alpha \in E$ and $E \backslash \alpha$ is an interval in γ. It turns out that such E

can be described using the reduction operation. In order to do this, take an arbitrary $F \in \mathcal{F}_k$ with $\alpha \in F$. Then:

$$\begin{aligned}
red_{\alpha+1}(F) &= F \cap (\alpha + 1) \cup [\alpha + 1, \, \alpha + 1 + |F \backslash (\alpha + 1)| \,) \\
&= (\alpha)_k^- \cup [\alpha, \, \alpha + 1 + (m_k - (\|\alpha\|_k + 1)) \,) \\
&= (\alpha)_k^- \cup [\alpha, \, \alpha + m_k - \|\alpha\| \,).
\end{aligned}$$

Due to the Remark 5.0.16, it is straightforward that $E = red_{\alpha+1}(F)$.

The reduction operation will be mainly used in the case where $p \in \mathbb{P}(\mathcal{F})$ and $\alpha = \gamma$. We now illustrate such situation for convenience of the reader.
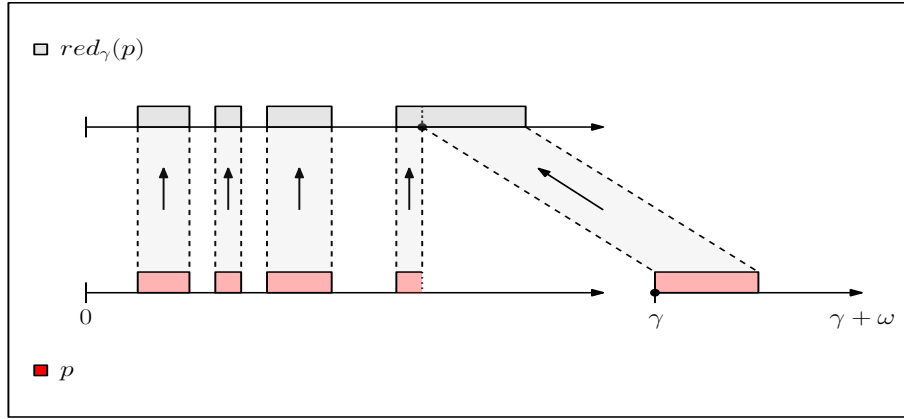
By arguing in a similar manner as in the Remark 5.1.4, we get the following lemma. The proof of it is left to the reader.

Lemma 5.1.5. *If $p \in FIN(\gamma + \omega)$ is such that $p \cap [\gamma, \gamma + \omega)$ is an initial segment of $[\gamma, \gamma + \omega)$, then the following conditions are equivalent:*

(a) $p \in \mathbb{P}(\mathcal{F})$,

(b) $red_\gamma(p) \in \mathcal{F}$.

Even though the previous lemma is useful, the main tool for defining and extending conditions in $\mathbb{P}(\mathcal{F})$ relies in the next definition.

Definition 5.1.6 (The cut operation). Let $F \in \mathrm{FIN}(\gamma)$ and $\alpha \in \gamma$. We define the *cut of F at α* as follows:

$$Cut_\alpha(F) = (F \cap \alpha) \cup [\gamma, \gamma + |F \backslash \alpha|)$$

The following picture illustrates the previous definition for the particular case in which $\alpha \in F$. This is the main situation in which the cut operation will be applied.
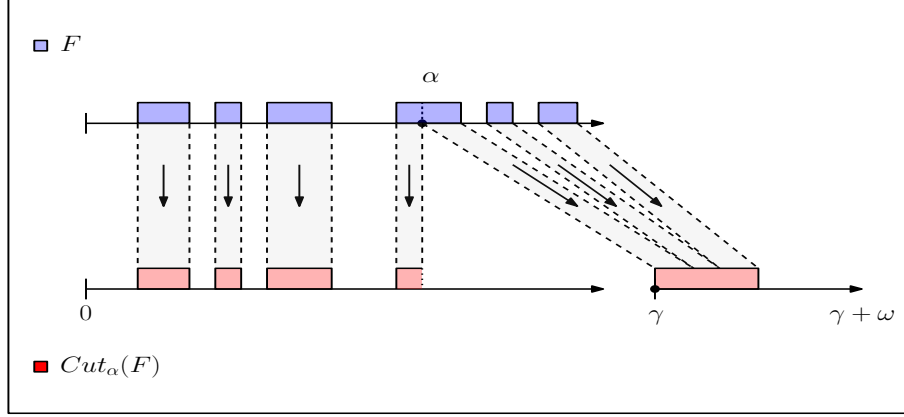
Remark 5.1.7. Suppose that $F \in \mathcal{F}$ and $\alpha \in \gamma$. If $\phi : F \longrightarrow Cut_\alpha(F)$ is the increasing bijection, then

$$\mathcal{F}(Cut_\alpha(F)) = \{\, \phi[H] \, : \, H \in \mathcal{F}(F) \,\}$$

due to the Remark 5.0.10. In particular, this means that $Cut_\alpha(F_i)$ is the *ith* element of the canonical decomposition of $Cut_\alpha(F)$ for each $i < n_{\rho^F}$. In other words, $Cut_\alpha(F_i) = (Cut_\alpha(F))_i$. On the other hand, it is not necessarily true that $\phi[H] = Cut_\alpha(H)$ for a given $H \in \mathcal{F}(F)$. A particular case in which the previous equality holds is when $\alpha \in H$ and $H\backslash\alpha$ is an interval in F (Thus, an initial segment of $F\backslash\alpha$).

Remark 5.1.8. The reduction and cut operations are in some sense inverses of each other. That is, if $p \in \mathbb{P}(\mathcal{F})$ and $p \cap \gamma \neq \emptyset$, then $Cut_{\alpha_p+1}(red_\gamma(p)) = p$. On the other hand, if $F \in \text{FIN}(\gamma)$ and $\alpha \in F$ then $red_\gamma(Cut_{\alpha+1}(F)) = F$.

Lemma 5.1.9. *Let $F \in \mathcal{F}$ and $\alpha \in \gamma$. Then $Cut_\alpha(F) \in \mathbb{P}(\mathcal{F})$. Furthermore, if $F, G \in \mathcal{F}$ are such that $F \subseteq G$ and $\alpha \in F$ then $Cut_\alpha(G) \leqslant Cut_\alpha(F)$.*

Proof. For the first part of the lemma, suppose that $F \in \mathcal{F}$ and $\alpha \in \gamma$. Let $k = \rho^F$. Then $m_k = |F| = |Cut_\alpha(F)|$. In this way, $Cut_\alpha(F)$ satisfies the condition (I) of Definition 5.1.1. Now, $Cut_\alpha(F) \cap \gamma = F \cap \alpha \sqsubseteq F$. Therefore, the condition (II) of the same definition holds for $Cut_\alpha(F)$. Finally, $Cut_\alpha(F) \cap [\gamma, \gamma + \omega) = [\gamma, \gamma + |F\backslash\alpha|)$. Thus, $Cut_\alpha(F)$ satisfies the condition (III) of Definition 5.1.1, which means that $Cut_\alpha(F) \in \mathbb{P}(\mathcal{F})$.

Next, we prove the second part of the lemma. Let $F, G \in \mathcal{F}$ be such that $F \subseteq G$ and let $\alpha \in F$. If $F = G$, the result is obvious. So we may assume that the inclusion between F and G is proper. It follows that $k < l$ where $k = \rho^F$ and $l = \rho^G$. Therefore, we are in the conditions of applying the lemma 5.0.14 to α, G and k. In this way, we get $E \in \mathcal{F}_k$ for which $\alpha \in E$, $E \subseteq G$ and $E\backslash\alpha$ is an interval in G. Note that $|E\backslash\alpha| = |F\backslash\alpha|$ and $F \cap \alpha = (\alpha)_k^- = E \cap \alpha$. Thus, $Cut_\alpha(F) = Cut_\alpha(E)$. Now, let $\phi : G \longrightarrow Cut_\alpha(G)$ be the increasing bijection. Due to the Remark 5.1.7,

$$\mathcal{F}(Cut_\alpha(G)) = \{\, \phi[H] \, : \, H \in \mathcal{F}(G) \,\}.$$

Finally, just note that $\phi[E] = Cut_\alpha(E)$ by virtue of the same remark. ∎

Lemma 5.1.10. *Let $k \in \omega$, $p \in \mathbb{P}_k(\mathcal{F})$ (that is, $k = k_p$) and $\alpha \in \gamma$ be such that $(\alpha)_k^- = p \cap \gamma$. Then $Cut_\alpha(G) \leqslant p$ for each $G \in \mathcal{F}$ with $\alpha \in G$ and $\rho^G \geqslant k$.*

Proof. Let $G \in \mathcal{F}$ be as in the hypotheses. Consider $F \in \mathcal{F}_k$ for which $\alpha \in F$ and $F \subseteq G$. According to the Lemma 5.1.9, $Cut_\alpha(G) \leqslant Cut_\alpha(F)$. To finish, just note that $|Cut_\alpha(F)| = |p|$ and $Cut_\alpha(G) \cap \gamma = G \cap \alpha = (\alpha)_k^- = p \cap \gamma$. In this way, $Cut_\alpha(F) = p$ due to the condition (III) of Definition 5.1.1. $\hfill\blacksquare$

5.1.1 Construction schemes in ZFC

The next definition appeared for the first time in [112]. This hypothesis is sufficient to build construction schemes in a recursive manner.

Definition 5.1.11 (The IH_1 property)**.** Let $A \in \mathrm{FIN}(\gamma)$, $\alpha \in \gamma$ and $F \in \mathcal{F}$. We say that $IH_1(\alpha, A, F)$ holds if:

(1) $A \subseteq F_0$,

(2) $R(F) = F \cap \alpha$.

Additionally, we say that $\mathcal{F}$ satisfies IH_1 if for all $A \in \mathrm{FIN}(\gamma)$ and $\alpha \in \gamma$, there is $F \in \mathcal{F}$ for which $IH_1(\alpha, A, F)$ holds.

Proposition 5.1.12. *Suppose that τ is a good type. Then $\mathcal{F}(\omega)$ satisfies IH_1.*

Proof. Let $A \in \mathrm{FIN}(\gamma)$ and $\alpha \in \omega$. Since τ is a good type, there is $k > \max(A)$ for which $r_{k+1} = \alpha$. Let $F = m_{k+1}$. Then $F \in \mathcal{F}(\omega)$, $F_0 = m_k \supseteq A$ and $R(F) = r_{k+1} = \alpha$. This finishes the proof. $\hfill\blacksquare$

The following lemma is easy.

Lemma 5.1.13. *Suppose that γ is a limit of limit ordinals and $\mathcal{F}|_\delta$ satisfies IH_1 for each limit $\delta < \gamma$. Then $\mathcal{F}$ also satisfies IH_1.*

Definition 5.1.14. Given a filter $\mathcal{G}$ over $\mathbb{P}(\mathcal{F})$, we define $\mathcal{F}^{\mathcal{G}}$ as $\bigcup_{p \in \mathcal{G}} \mathcal{F}(p)$. Finally, $\mathcal{F}^{Gen}$ denotes the name for $\mathcal{F}^{\mathcal{G}}$ where G is a generic filter.

Lemma 5.1.15. *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. Then the set $\{\, p \in \mathbb{P}(\mathcal{F}) \,:\, p \cap \gamma \neq \emptyset \,\}$ is open dense in $\mathbb{P}(\mathcal{F})$.*

Proof. Let $q \in \mathbb{P}(\mathcal{F})$. Without loss of generality we can assume that $q \neq \emptyset$. We need to find $p \leqslant q$ such that $p \cap \gamma \neq \emptyset$. If $q \cap \gamma \neq \emptyset$, we are done. So suppose that $q \cap \gamma = \emptyset$. Then $q = [\gamma, \gamma + m_{k_q})$ due to the condition (III) of Definition 5.1.1. Since τ is a good type, then there is $k > k_q$ such that $r_{k+1} = 0$. Let $F = m_{k+1}$ and $\alpha = m_k$. Then $F \in \mathcal{F}(\omega) \subseteq \mathcal{F}$. By means of the 5.1.9,

$$m_k \cup [\gamma, \gamma + m_k + 1 - m_k) = Cut_\alpha(F) \in \mathbb{P}(\mathcal{F}).$$

Let $p = Cut_\alpha(F)$. Note that the second piece of the canonical decomposition of p, that is p_1, is equal to $[\gamma, \gamma + m_k)$. From this, it is easy to see that $[\gamma, \gamma + m_j) \in \mathcal{F}(p_1)$ for each $j \leqslant k$. In particular, $q \in \mathcal{F}(p_1) \subseteq \mathcal{F}(p)$. In other words, $p \leqslant q$. Since $p \cap \gamma = m_k \neq \emptyset$, the proof is over. $\hfill\blacksquare$

Lemma 5.1.16. *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. Then the set $\{\, p \in \mathbb{P}(\mathcal{F}) \,:\, p \cap [\gamma, \gamma + \omega) \neq \emptyset \,\}$ is open dense in $\mathbb{P}(\mathcal{F})$.*

Proof. Let $q \in \mathbb{P}(\mathcal{F})$. Without loss of generality we may assume that $q \neq \emptyset$. We need to find $p \leqslant q$ with $p \backslash \gamma \neq \emptyset$. If $q \backslash \gamma \neq \emptyset$, there is nothing to do. In this way, we may assume that $q \subseteq \gamma$. In this case, it follows that $q \in \mathcal{F}$. Let $\alpha = \alpha_p + 1 = \max(q) + 1$. Since $\mathcal{F}$ satisfies IH_1, there is $F \in \mathcal{F}$ with $q \cup \{\alpha\} \subseteq F_0$ and $\alpha \cap F = R(F)$. Note that q is actually a subset of $R(F)$. Let $p = Cut_\alpha(F)$. According to the Lemma 5.1.9, $p = Cut_\alpha(F) \leqslant Cut_\alpha(q) = q$. Furthermore, $p \backslash \gamma \neq \emptyset$. This finishes the proof. ■

Lemma 5.1.17. *Suppose that $\mathcal{F}$ is a construction scheme over γ satisfying IH_1. For any $\alpha \in \gamma + \omega$, the set*

$$\mathcal{D}'_\alpha = \{p \in \mathbb{P}(\mathcal{F}) \,:\, \alpha \in p\}$$

is open dense in $\mathbb{P}(\mathcal{F})$.

Proof. Let $q \in \mathbb{P}(\mathcal{F})$. By Lemmas 5.1.15 and 5.1.16, we may assume that $q \cap \gamma \neq \emptyset$ and $q \backslash \gamma \neq \emptyset$. In this way, α_q is well defined and $\alpha_q + 1 \in F$ where $F = red_\gamma(q)$. In particular, $(\alpha_q + 1)^-_{k_q} = q \cap \gamma$. Recall that our goal is to find $p \in \mathbb{P}(\mathcal{F})$ with $p \leqslant q$ and such that $\alpha \in p$. The proof of this fact is divided into two cases.

<u>Case 1</u>: If $\alpha < \gamma$.

Proof of case. Since $\mathcal{F}$ satisfies IH_1, there is $G \in \mathcal{F}$ so that $\{\alpha\} \cup F \subseteq G_0$ and $(\alpha_q + 1) \cap G = R(G)$. Let $k = \rho^G$ and $\beta = \min(G_1 \backslash R(G))$. Note that $k > k_q$ because $F \subseteq G_0$. Moreover, $(\beta)^-_{k-1} = R(F) = (\alpha_q + 1)^-_{k-1}$. Since $k_q \leqslant k$, it follows that $(\beta)^-_{k_q} = (\alpha_q + 1)^-_{k_q} = q \cap \gamma$. Let $p = Cut_\beta(G)$. According to the Lemma 5.1.10, $p \leqslant q$. Moreover, $p \cap \gamma = G_0$ and this set contains α. Hence, the proof of this case is done. □

<u>Case 2</u>: If $\alpha \geqslant \gamma$.

Proof of case. Let $n = |\alpha \backslash \gamma| + 1$. Again, using that $\mathcal{F}$ satisfies IH_1, we can get $G \in \mathcal{F}$ so that $F \cup [\alpha_q, \alpha_q + n] \subseteq G_0$ and $(\alpha_q + 1) \cap G = R(G)$. By arguing in a similar way as we did in the first case, we may conclude that for $\beta = \min(G_1 \backslash R(G))$, $p = Cut_\beta(G)$ is a condition in $\mathbb{P}(\mathcal{F})$ which is smaller than q. In order to finish, note that $[\alpha_q, \alpha_q + n] \subseteq G_0 \backslash R(G)$. Therefore, $n \leqslant |G_0 \backslash R(G)| = |G_1 \backslash R(G)|$. According to the definition of $Cut_\beta(G)$, we get that $|p \cap [\gamma, \gamma + \omega)| \geqslant n$. Thus, $\alpha \in [\gamma, \gamma + n] \subseteq p$. □

■

Lemma 5.1.18. *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. For any $F \in \mathcal{F}$, the set*

$$\mathcal{D}_F = \{\, p \in \mathbb{P}(\mathcal{F}) \,:\, F \in \mathcal{F}(p) \,\}$$

is open dense in $\mathbb{P}(\mathcal{F})$.

Proof. Let $q \in \mathcal{F}$. By applying Lemma 5.1.17 to each element of F, we get that there is $p \in \mathbb{P}(\mathcal{F})$ so that $p \leqslant q$ and $F \subseteq p$. According to the Lemma 5.1.9, $Cut_{\alpha_p+1}(red_\gamma(p)) \leqslant Cut_{\alpha_p+1}(F)$. That is, $Cut_{\alpha_p+1}(F) \in \mathcal{F}(Cut_{\alpha_p+1}(red_\gamma(p)))$. Note that $Cut_{\alpha_p+1}(F) = F$ because $F \subseteq \alpha_p + 1$. Moreover, $Cut_{\alpha_p+1}(red_\gamma(p)) = p$ by virtue of the Remark 5.1.8. This finishes the proof.

∎

Lemma 5.1.19. *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. For any $A \in FIN(\gamma + \omega)$ and each $\alpha \in \gamma + \omega$, the set*

$$\mathcal{E}_{\alpha,A} = \{\, p \in \mathbb{P}(\mathcal{F}) \,:\, \alpha \in p, \ A \subseteq p_0, \ \text{and} \ \alpha \cap p = R(p) \,\}$$

is dense in $\mathbb{P}(\mathcal{F})$.

Proof. Let $q \in \mathbb{P}$. By means of the previous lemmas, we can assume without loss of generality that $q \cap \gamma \neq \emptyset$, $q \backslash \gamma \neq \emptyset$ and $\{\alpha\} \cup A \subseteq q$. Let $\alpha' = red_\gamma(q)(|\alpha \cap q|)$. That is, α' is the element of $red_\gamma(q)$ which corresponds to α via the increasing bijection from $red_\gamma(q)$ to q. Since $\mathcal{F}$ satisfies IH_1, there is $F \in \mathcal{F}$ so that $red_\gamma(q) \subseteq F_0$ and $F \cap \alpha' = R(F)$. Let $p = Cut_{\alpha_q+1}(F)$. We claim that p is an element of $\mathcal{E}_{\alpha,A}$ which is smaller that q. Indeed, since $\alpha_p + 1 \in red_\gamma(q)$, then $(\alpha_q + 1)_{k_q}^- = q \cap \gamma$. Furthermore, $\rho^F > k_q$ because F_0 contains the reduction of q to γ. In this way, $p \leqslant q$ by virtue of the Lemma 5.1.10. Moreover, by virtue of the Remarks 5.1.7 and 5.1.8, the following chain of inclusions hold:

$$A \subseteq q = Cut_{\alpha_q+1}(red_\gamma(q)) \subseteq Cut_{\alpha_q+1}(F_0) = p_0.$$

Now, let $\phi : F \longrightarrow Cut_{\alpha_q+1}(F)$ be the increasing bijection. Note that $red_\gamma(q) \backslash \alpha_q$ is an initial segment of F. According to the last part of the Remark 5.1.7, $\phi[red_\gamma(q)] = Cut_{\alpha_q+1}(red_\gamma(q)) = q$. In particular, $\phi[\alpha'] = \alpha$. Since $\phi[R(F)] = R(p)$, we have that $R(p) = \phi[\alpha' \cap F] = \alpha \cap p$. ∎

Proposition 5.1.20. *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. Let $\mathcal{G}$ be a filter over $\mathbb{P}(\mathcal{F})$ intersecting $\mathcal{D}'_\alpha$, $\mathcal{D}_F$ and $\mathcal{E}_{\alpha,A}$ for all $\alpha \in \omega + \gamma$, $A \in FIN(\omega + \gamma)$ and $F \in \mathcal{F}^1$. Then $\mathcal{F}^\mathcal{G}$ is a construction scheme over $\gamma + \omega$ which contains $\mathcal{F}$ and satisfies IH_1.*

Proof. If we use that $\mathcal{G}$ intersects each $\mathcal{D}'_\alpha$, and argue in a similar fashion as we did in Lemma 5.0.13, it is easy to see that $\mathcal{F}^G$ is a construction scheme over $\gamma + \omega$. Furthermore, $\mathcal{F} \subseteq \mathcal{F}^\mathcal{G}$ because $\mathcal{G}$ intersects each $\mathcal{D}_\mathcal{F}$, and $\mathcal{F}^\mathcal{G}$ satisfies IH_1 because $\mathcal{G}$ intersects each $\mathcal{E}_{\alpha,A}$. ∎

Suppose that τ is a good type, By means of the results 5.1.12, 5.1.20, 5.0.13, and 5.1.13, we can recursively construct, for each limit $\gamma < \omega_1$, a construction scheme $\mathcal{F}^\gamma$ which satisfies IH_1 and such that $\mathcal{F}^\delta \subseteq \mathcal{F}^\gamma$ whenever $\delta < \gamma$. If we use one more time the Lemmas 5.0.13 and 5.1.13, we can conclude that

$$\mathcal{F} = \bigcup_{\gamma \in Lim} \mathcal{F}^\gamma$$

---

[1]Such $\mathcal{G}$ exists due to the Rasiowa-Sikorski Lemma

is a construction scheme over ω_1 which satisfies IH_1. Thus, we have proved Theorem 2.2.5.

Theorem 2.2.5. *For any good type there is a construction scheme over ω_1 of that type which satisfies IH_1.*

5.2 Diamond principle and FCA(part)

The purpose of this section is to prove that Jensen's $\Diamond$-principle implies FCA. In order to do this, we will work in the same manner as we did in the Subsection 5.1.1. That is, we want to find a suitable property IH_2 which is satisfied by $\mathcal{F}(\omega)$. Furthermore, given a construction scheme $\mathcal{F}$ over a limit ordinal $\gamma \leqslant \omega_1$, we want the two following things to happen:

(a) If γ is a limit of limit ordinals and $\mathcal{F}|_\delta$ satisfies IH_2 for each limit $\delta < \gamma$, then $\mathcal{F}$ also satisfies IH_2.

(b) If γ is countable and $\mathcal{F}$ satisfies IH_1 and IH_2 then there is a construction scheme $\mathcal{F}'$ over $\gamma + \omega$ containing $\mathcal{F}$ which satisfies IH_1 and IH_2.

By doing this, we may conclude that there is a construction scheme over ω_1 which satisfies IH_2. Finally, we want that:

(c) Any construction scheme over ω_1 which satisfies IH_2 is fully capturing.

While the points (a) and (c) are relatively easy to guarantee, it turns out that finding a property IH_2 which also satisfies the point (b) is a non-trivial problem.

For the rest of this section, we fix a countable limit ordinal γ and a construction scheme $\mathcal{F}$ be over γ of type τ. Moreover, we fix a $\Diamond$-sequence $\langle D_\alpha \rangle_{\alpha \in \mathrm{Lim}}$.

Definition 5.2.1 (Block interval sequence). Let $j \in \omega$ and A be a non-empty subset of ω_1. We say that a sequence $\mathbb{I} = \langle \mathbb{I}(i) \rangle_{i<j} \subseteq \mathrm{FIN}(A)$ is a *block interval sequence over A* if:

■ For any $i < t$, $\mathbb{I}(i)$ is an interval in A.

■ For all $i < i' < t$, $\mathbb{I}(i) < \mathbb{I}(i')$.

Given $\alpha \in \omega_1$, we let $Bl(\alpha, A)$ to be the set of all interval sequences over $A \backslash \alpha$.[2] Given $\alpha \in X$, we denote $Bl(\|\alpha\|_k, m_k)$ simply as $Bl_k(\alpha)$ for each $k \in \omega$.

Remark 5.2.2. Note that a block interval sequence $\mathbb{I}$ may be empty even though its elements are not. This case is not pathological, and will be highly relevant for the proof of Theorem 5.2.16.

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[2]Note that $Bl(\alpha, A) = Bl(0, A \backslash \alpha)$ for all $\alpha \in \omega_1$ and $A \in \mathrm{FIN}(\omega_1)$.

Definition 5.2.3. Suppose that A and B are non-empty subsets of ω_1 and $\phi : A \longrightarrow B$ is an increasing function. Given $\mathbb{I} = \langle \mathbb{I}(i) \rangle_{i<j} \in Bl(0, A)$, we let

$$\phi \circ \mathbb{I} = \langle \, \phi[\mathbb{I}(i)] \, \rangle_{i<j}.$$

Recall that if B has size m for some $m \in \omega$, then we identity the increasing bijection $\phi : m \longrightarrow B$ with B, and we denote the inverse of such function as B^{-1}. Following this convention, if $A = m$ and $\mathbb{I}$ is as before, $\phi \circ \mathbb{I}$ is denoted as $B \circ \mathbb{I}$. Analogously, if $\mathbb{J} \in Bl(0, B)$, we denote $\phi^{-1} \circ \mathbb{J}$ as $B^{-1} \circ \mathbb{J}$.

Remark 5.2.4. Suppose that A, B, ϕ and $\mathbb{I}$ are as in the previous definition. Since ϕ is increasing, then $\phi \circ \mathbb{I}(i) < \phi \circ \mathbb{I}(i')$ for all $i < i < j$. However, it is not necessarily true that $\phi \circ \mathbb{I}(i)$ is an interval for a given i. A particular case in which this occurs is when $im(\phi)$ is itself an interval in B. In fact, it is easy to show that $im(\phi)$ is an interval in B if and only if

$$\{\phi \circ \mathbb{I} \, : \, \mathbb{I} \in Bl(0, A)\} \subseteq Bl(0, B).$$

Moreover, if ϕ is surjective then the equality between the previous sets will hold. Particularly, if $\alpha \in \gamma$ and $k \in \omega$, then

$$Bl(\alpha, F) = \{ \, F \circ \mathbb{I} \, : \, \mathbb{I} \in Bl_k(\alpha) \, \},$$

$$Bl_k(\alpha) = \{ F^{-1} \circ \mathbb{I} \, : \, \mathbb{I} \in Bl(\alpha, F) \, \}$$

for all $F \in \mathcal{F}_k$ with $\alpha \in F$.
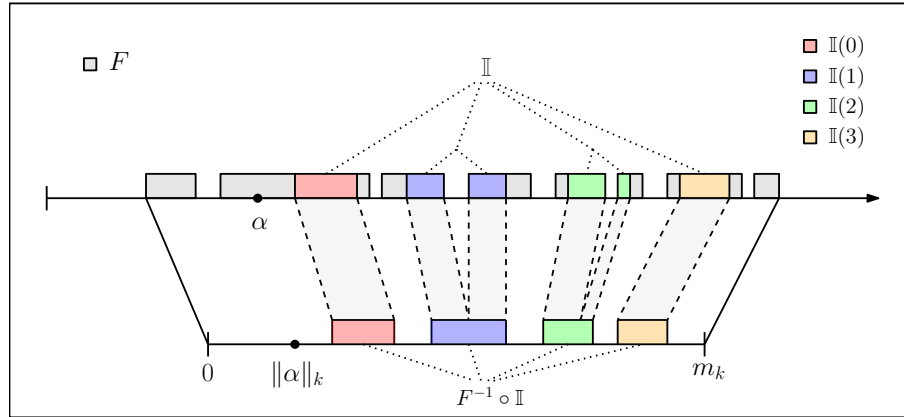
In here, F represents an element of $\mathcal{F}_k$ with $\alpha \in F$ and $\mathbb{I}$ is an element of $Bl(\alpha, F)$.

Definition 5.2.5 (Good pairs and good sets)**.** Let $\alpha \in \omega_1$ and $A \in \text{FIN}(\omega_1)$. we say that a pair $(\mathbb{I}, z)$ is (α, A)-*good* if the following conditions hold:

- $z \in A \backslash \alpha$,

- $\mathbb{I} \in Bl(\alpha, A \cap (z+1))$. That is, $\mathbb{I} \in Bl(\alpha, A)$ and either $\mathbb{I} = \emptyset$ or $\max(\bigcup \mathbb{I}) \leqslant z$.

A non-empty set of ordered pairs T is said to be (α, A)-*Good* if every $(\mathbb{I}, z) \in T$ is (α, A)-good. We denote the family of (α, A)-Good sets as $Good(\alpha, A)$. Given $\alpha \in \gamma$ and $k \in \omega$, we denote $Good(\|\alpha\|_k, m_k)$ as $Good_k(\alpha)$.
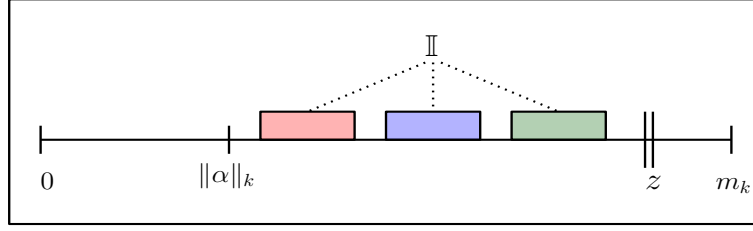
In here, $(\mathbb{I}, z)$ represents an $(\|\alpha\|_k, m_k)$-good pair.

Definition 5.2.6. Suppose that A and B are non-empty subsets of ω_1 and $\phi : A \longrightarrow B$ is an increasing function. Given $\alpha \in \omega_1$ and $T \in Good(\alpha, A)$ we define

$$\phi \bullet T = \{ (\phi \circ \mathbb{I}, \phi(z)) \, : \, (\mathbb{I}, z) \in T \}.$$

In the particular case where $|B| = m$ for some $m \in \omega$ and $A = m$, we denote $\phi \bullet T$ as $B \bullet T$. Analogously, if $T' \in Good(\alpha, B)$ then $\phi^{-1} \bullet T'$ is denoted as $B^{-1} \bullet T'$.

Remark 5.2.7. If $\alpha \in \gamma$ and $k \in \omega$ then

$$Good(\alpha, F) = \{ F \bullet T \, : \, T \in Good_k(\alpha) \},$$
$$Good_k(\alpha) = \{ F^{-1} \bullet T \, : \, T \in Good(\alpha, F) \}$$

for each $F \in \mathcal{F}_k$ with $\alpha \in F$.

For the next the definition, it is convenient to recall the function ϕ_l^β that was presented in Definition 5.0.22.

Definition 5.2.8. Let $\beta \in \gamma$ and $k \leqslant l \in \omega$. We define $\pi_\beta^{k,l} : \|\alpha\|_k + 1 \longrightarrow \|\beta\|_l + 1$ as:

$$\pi_\beta^{k,l} = \phi_l^\beta \circ (\phi_k^\beta)^{-1}.$$

It is not hard to see that $\pi_\alpha^{k,l}$ is just the increasing bijection from $\|\alpha\|_k + 1$ to $(\|\alpha\|_l)_k$.

Definition 5.2.9 (The $\bigstar$ relation). Let $\beta, \xi \in \gamma$, $2 \leqslant k < l \in \omega$, and $T \in Good_k(\beta)$. We say that T *guesses* (β, ξ, k, l), and write it as $T\bigstar(\beta, \xi, k, l)$, if there is $(\mathbb{I}, z) \in T$ for which:

(a) $\|\beta\|_l \leqslant \|\xi\|_l$ and $\|\xi\|_k = z$,

(b) $(\xi)_k \circ \mathbb{I} \in Bl(0, (\xi)_l)$. Equivalently, if $\pi_\xi^{k,l} \circ \mathbb{I} \in Bl(0, m_l)$,

(c) $(\xi)_l(\|\beta\|_l) \in (\xi)_k$. Equivalently, $\|\beta\|_l \in (\|\xi\|_l)_k$.
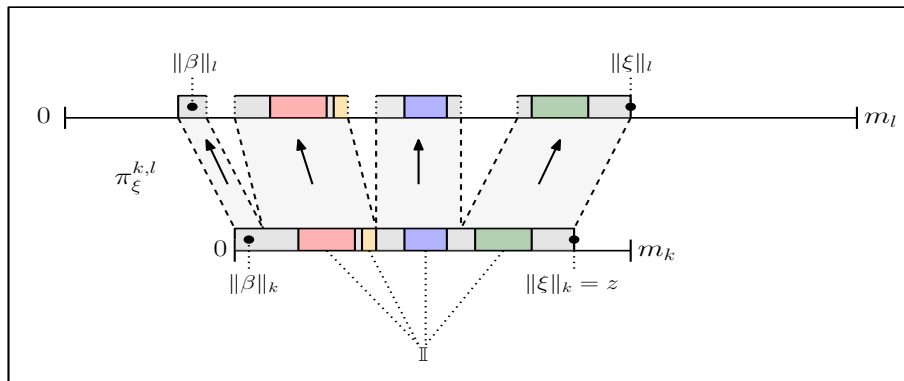
In here, the pair $(\mathbb{I}, z)$ represent an element of T which testifies that
$T \star (\beta, \xi, k, l)$.

Remark 5.2.10. If $T\bigstar(\beta, \xi, k, l)$ and ξ' is such that $\|\xi'\|_l = \|\xi\|_l$ then $T\bigstar(\beta, \xi', k, l)$.

Remark 5.2.11. The previous definition will often be applied in cases where $\xi < \beta$.

Recall that we fixed $\langle D_\delta \rangle_{\delta \in \text{Lim}}$ a $\Diamond$-sequence at the start of this section.

Definition 5.2.12 (The $\checkmark$ relation)**.** Let $\beta \in \gamma$, $k \leqslant l \in \omega$ and $\delta \in Lim$. Given $T \in Good_k(\beta)$ and $C \in \text{FIN}(\gamma)$, we say that (C, δ) *accepts* (T, β, k, l), and write it as $(C, \delta)\checkmark(T, \beta, k, l)$, if the following conditions hold:

(1) $C \subseteq D_\delta$ and C is captured at level l,

(2) $T\bigstar(\beta, C(0), k, l-1)$,.

(3) $|(\beta)_{l-1} \cap \delta| = r_l$. Equivalently, if $F \in \mathcal{F}_l$ is such that $C \subseteq F$, then $(\beta)_{l-1} \cap \delta = R(F)$.

Whenever there is C for which $(C, \delta)\checkmark(T, \beta, k, l)$, such C has cardinality at least 1 and at most n_l due to the point (1). Thus, we can define $j(\delta, T, \beta, k, l)$ as the maximum of such cardinalities. That is:

$$j(\delta, T, \beta, k, l) = \max(\, j \leqslant n_l \,:\, \exists C \in [\gamma]^j \,(\,(C, \delta)\checkmark(T, \beta, k, l)\,)\,).$$

If there is no such C, then we define $j(\delta, T, \beta, k, l)$ as 0.

Definition 5.2.13 (The IH_2 property)**.** Given $\delta \in \text{Lim} \cap \gamma$, we say that $IH_2(\delta, \mathcal{F})$ holds if one of the two following mutually excluding conditions occurs:

(A) There are infinitely many $l \in \omega$ for which there is $C \in \text{FIN}(D_\delta)$ which is fully captured at level l. In this case, we will say that $IH_2^A(\delta, \mathcal{F})$ holds.

(B) For all $\beta \in [\delta, \gamma)$, $2 \leqslant k \in \omega$ and $T \in Good_k(\beta)$ there are infinitely many $k \leqslant l \in \omega$ for which $j = j(\delta, T, \beta, k, l) < n_l$ and such that:

(B.1) $|(\beta)_{l-1} \cap \delta| = r_l$,
(B.2) $\Xi_\beta(l) = j$,
(B.3) If $j > 0$ then there is $C \in [\delta]^j$ for which $(C, \delta)\checkmark(T, \beta, k, l)$ and $C \subseteq (\beta)_l$.

In this case we will say that $IH_2^B(\delta, \mathcal{F})$ holds.

Finally, we say that $\mathcal{F}$ satisfies IH_2 if $IH_2(\delta, \mathcal{F})$ holds for any $\delta \in \text{Lim} \cap \gamma$.

The two following results follow directly from the definition.

Proposition 5.2.14. $\mathcal{F}(\omega)$ *satisfies* IH_2.

Lemma 5.2.15. *Assume that γ is a limit of limit ordinals and for each limit $\gamma' < \gamma$, $\mathcal{F}|_{\gamma'}$ satisfies IH_2. Then $\mathcal{F}$ satisfies IH_2.*

More important, the property IH_2 in fact implies full capturing when $\gamma = \omega_1$.

Theorem 5.2.16. *Suppose that $\gamma = \omega_1$ and $\mathcal{F}$ satisfies IH_2. Then $\mathcal{F}$ is fully capturing.*

Proof. We will prove the Theorem by appealing to the equivalence of fully capturing stated in Lemma 2.4.15. Let $S \in [\omega_1]^{\omega_1}$. Since $\langle D_\delta \rangle_{\delta \in \mathrm{Lim}}$ is a $\Diamond$-sequence, there is $\delta \in \mathrm{Lim}$ so that:

1. $S \cap \delta = \delta$.

2. $(\delta, <, D_\delta, \mathcal{F}|_\delta)$ is an elementary submodel of $(\omega_1, <, S, \mathcal{F})$.

Suppose towards a contradiction that there is no $C \in \mathrm{FIN}(S)$ so that C is fully captured. By elementary, the same is true for D_δ. In other words, $IH_2^A(\delta, \mathcal{F})$ fails and consequently $IH_2^B(\delta, \mathcal{F})$ holds. Let us fix $\beta \in S \backslash \delta$ and $k = 2$. We now define $\mathbb{I} = \emptyset$, $z = \|\beta\|_2$ and $T = \{(\mathbb{I}, z)\} \in Good_2(\beta)$. According to the point (B) in Definition 5.2.13, there is $2 \leqslant l \in \omega$ for which $j = j(\delta, T, \beta, 2, l) < n_l$ and such that:

- $|(\beta)_{l-1} \cap \delta| = r_l$,

- $\Xi_\beta(l) = j$,

- If $j > 0$ then there is $C \in [\delta]^j$ for which $(C, \delta)\checkmark(T, \beta, 2, l)$ and $C \subseteq (\beta)_l$.

We will arrive to the desired contradiction by finding a finite set $C' \in \mathrm{FIN}(\omega_1)$ for which $(C', \delta)\checkmark(T, \beta, 2, l)$ and $|C'| = j + 1$. The proof of this is divided into two cases:

<u>Case 1</u>: If $j = 0$.

Proof of case. In this case, as $\Xi_\beta(l) = 0$, it is straightforward that $\{\beta\}$ is captured at level l. By elementarity, there is $\xi \in D_\delta$ so that $\|\xi\|_l = \|\beta\|_l$. In particular, $\{\xi\}$ is captured at level l. The proof of this case ends by proving the following claim.

<u>Claim 1</u>: $(\{\xi\}, \delta)\checkmark(T, \beta, k, l)$.

Proof of claim. It is enough to show that $T\bigstar(\beta, \xi, 2, l-1)$. For this, note that the point (a) of Definition 5.2.9 is satisfied for the unique pair $(\mathbb{I}, z) \in T$ because $\|\beta\|_l = \|\xi\|_l$. In particular, $\Delta(\beta, \xi) > l$ so $\|\xi\|_2 = \|\beta\|_2 = z$. The point (b) of Definition 5.2.9 holds because $\mathbb{I} = \emptyset$. Therefore, $\pi_\xi^{2,l} = \emptyset \in Bl(0, m_l)$. Finally, the point (c) of Definition 5.2.9 is satisfied because $(\xi)_l(\|\beta\|_l) = (\xi)_l(\|\xi\|_l) = \xi$. $\qquad\square$

$\square$

<u>Case 2</u>: If $j > 0$.

Proof of case. Let $C \in [\delta]^j$ be such that $(C, \delta)\checkmark(T, \beta, 2, l)$ and $C \subseteq (\beta)_l$. By elementarity, there is $\xi \in D_\delta$ so that $\|\xi\|_l = \|\beta\|_l$ and $C \subseteq (\xi)_l$. Particularly, $\Xi_\xi(l) = j$ and $\rho(C(i), \xi) \leqslant l$ for each $i < j$. In fact, as $\Xi_{C(i)}(l) = i$ because C is captured at level l, we have that $\rho(C(i), \xi) = l$ due to Lemma 2.3.9. We will finish the proof of this case by showing the next claim.

<u>Claim 2</u>: $(C \cup \{\xi\}, \delta)\checkmark(T, \beta, k, l)$.

Proof of claim. It suffices to prove that $C \cup \{\xi\}$ is captured at level l. For this, we appeal to the Proposition 2.4.10. By the previous observations, we only need to prove that $\Delta(C(i), \xi) = \rho(C(i), \xi)$ for each $i < j$. First note that, according to the point (2) in Definition 5.2.12, $T \bigstar (\beta, C(0), 2, l-1)$. As $(\emptyset, \|\beta\|_2)$ is the only element of T, then $\|\beta\|_{l-1} \leqslant \|C(0)\|_{l-1}$, $\|C(0)\|_2 = \|\beta\|_2$ and $\|\beta\|_{l-1} \in (\|C(0)\|_{l-1})_2$. In other words, $\|\beta\|_{l-1}$ is in the domain of $\phi_2^{\|C(0)\|_{l-1}}$. Moreover,

$$\phi_2^{\|C(0)\|_{l-1}}(\|C(0)\|_{l-1}) = \| \, \|C(0)\|_{l-1} \, \|_2 = \|C(0)\|_2$$
$$= \|\beta\|_2 = \| \, \|\beta\|_{l-1} \, \|_2 = \phi_2^{\|C(0)\|_{l-1}}(\|\beta\|_{l-1})$$

due to the part (a) of Proposition 5.0.23. Since $\phi_2^{\|C(0)\|_l}$ is one to one, it follows that $\|C(0)\|_{l-1} = \|\beta\|_{l-1}$. Now, as $\|\beta\|_l = \|\xi\|_l$ and $\|C(0)\|_{l-1} = \|C(i)\|_{l-1}$, then $\|\xi\|_{l-1} = \|C(i)\|_{l-1}$. That is, $l \leqslant \Delta(C(i), \xi) \leqslant \rho(C(i), \xi) \leqslant l$. This finishes the proof. $\qquad\square$

$\hfill\square$

$\hfill\blacksquare$

Before starting the next subsection, we present the key concept needed for proving Theorem 5.2.26.

Definition 5.2.17 (Transferring Good sets). Let $\beta \in \gamma$, $k \in \omega$ and $T \in Good_k(\beta)$. Given $k' > k$, we define $\mathcal{Q}_{k'}(\beta, T)$ as the set of all pairs $(F, \mathbb{I}) \in \mathcal{F}_k(m_{k'}) \times Bl_k(\beta)$ such that:
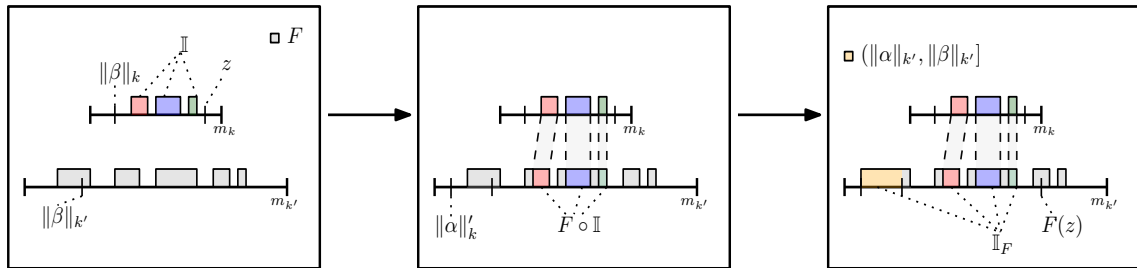
- There is $z \in m_k$ such that $(\mathbb{I}, z) \in T$,

- $\|\beta\|_{k'} \in F$,

- $F \circ \mathbb{I} \in Bl_{k'}(\beta)$. In other words, $F[\mathbb{I}(i)]$ is an interval in m'_k for every $i < \mathbb{I}$.

Given $\alpha \in (\beta)_{k'}^-($ and $(F, \mathbb{I}) \in \mathcal{Q}_{k'}(\beta, T)$, let $\mathbb{I}_F = (\|\alpha\|_{k'}, \|\beta, \|_{k'}]^\frown (F \circ \mathbb{I})$[3]. That is, $\mathbb{I}_F = \langle \mathbb{I}_F(i) \rangle_{i < \mathbb{I}+1}$ is given by:

$$\mathbb{I}_F(i) = \begin{cases} (\|\alpha\|_{k'}, \|\beta\|_{k'}] & \textit{if } i = 0 \\ F[\mathbb{I}(i-1)] & \textit{if } i > 0 \end{cases}$$

We define *the transferring of T relative to k', α and β* as:

$$\text{Transf}_{k'}(T, \alpha, \beta) = \{ \, (\mathbb{I}_F, F(z)) \, : \, (\mathbb{I}, z) \in T \textit{ and } (F, \mathbb{I}) \in \mathcal{Q}_{k'}(\beta, T) \, \}$$

In the first picture, we considered a a pair $(\mathbb{I}, z) \in T$ and $F \in \mathcal{F}_k(m_{k'})$ with $\|\beta\|_{k'} \in F$. In

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[3]In principle, $\mathbb{I}_F$ also depends on α. However, we will never use this concept with two different α's at the same time. For that reason, we omit any mention of that ordinal in the notation.

the second picture, we verified that $F \circ \mathbb{I}$ is an element of $Bl_{k'}(\beta)$. As this is the case, we can conclude that $(\mathbb{I}_F, F(z)) \in Transf_{k'}(T, \alpha, \beta)$.

Remark 5.2.18. Note that if $T \in Good_k(\beta)$, $\alpha \in (\beta)_k^-$ and $k' > k$ then $Transf_{k'}(T, \alpha, \beta) \in Good_{k'}(\alpha)$.

Lemma 5.2.19. *Let $\beta \in \gamma$, $k \in \omega$, $T \in Good_k(\beta)$, $k' > k$ and $\alpha \in (\beta)_{k'}^-$. Suppose that $l > k'$ and $\xi \in \gamma$ are such that:*

(1) $\|\beta\|_l \leqslant \|\xi\|_l$,

(2) $(\xi)_l(\|\beta\|_l) \in (\xi)_{k'}$. *In other words,* $\|\beta\|_l \in (\|\xi\|_l)_{k'}$

(3) $(\beta)_{k'}[(\|\alpha\|_{k'}, \|\beta\|_{k'}]] = (\beta)_{k'} \backslash (\alpha + 1)$ *is an interval in $(\beta)_l$.*

Then the following statements are equivalent:

(a) $T \bigstar (\beta, \xi, k, l)$.

(b) $Transf_{k'}(T, \alpha, \beta) \bigstar (\alpha, \xi, k', l)$.

Proof. Let $\mathcal{Q} = \mathcal{Q}_{k'}(\beta, T)$ and $T' = \mathrm{Transf}_{k'}(T, \alpha, \beta)$.

$(a) \Rightarrow (b)$. Let $(\mathbb{I}, z) \in T$ so that $\|\beta\|_k \leqslant \|\xi\|_k = z$, $\pi_\xi^{k,l} \circ \mathbb{I} \in Bl(0, m_l)$ and $\|\beta\|_l \in (\|\xi\|_l)_k$. Now, let us take $F \in \mathcal{F}_k(m_{k'})$ having $\|\xi\|_{k'}$ as an element. Before we continue, we will highlight some useful facts:

- $\|\xi\|_{k'} \in F$ so $F \cap (\xi + 1) = (\|\xi\|_{k'})_k$. Consequently, $F(a) = (\phi_k^{\|\xi\|_{k'}})^{-1}(a)$ for each $a \leqslant \|\xi\|_k$.

- $\|\xi\|_k = \phi_k^{(\|\xi\|_{k'})} \circ \phi_k^\xi(\xi)$ according to Lemma 5.0.24. Therefore, $F(\|\xi\|_k) = \|\xi\|_{k'}$ by virtue of the previous point.

- Again, by Lemma 5.0.24:
$$\phi_{k'}^{\|\xi\|_l} \circ \pi_\xi^{k,l} = \phi_{k'}^{\|\xi\|_l} \circ \phi_l^\xi \circ (\phi_k^\xi)^{-1} = \phi_{k'}^\xi \circ (\phi_k^\xi)^{-1} = (\phi_k^{\|\xi\|_{k'}})^{-1}.$$

<u>Claim 1</u>: $(F, \mathbb{I}) \in \mathcal{Q}$.

Proof of claim. First we will show that $\|\beta\|_{k'} \in F$. Indeed, since $k \leqslant k' < l$ and $\|\beta\|_l \in (\|\xi\|_l)_k$ then, by Proposition 5.0.23 and Lemma 5.0.24, we can conclude that

$$\|\beta\|_{k'} = \|\,\|\beta\|_l\,\|_{k'} = \phi_{k'}^{\|\xi\|_l}(\|\beta\|_l) \in \phi_{k'}^{\|\xi\|_l}[(\|\xi\|_l)_k]$$
$$= (\phi_{k'}^{\|\xi\|_l}(\|\xi\|_l))_k$$
$$= (\|\,\|\xi\|_l\,\|_{k'})_k = (\|\xi\|_{k'})_k \subseteq F.$$

Now, let $i < |\mathbb{I}|$. The claim will be over once we prove that $F[\mathbb{I}(i)]$ is an interval in $m_{k'}$. As $\mathbb{I}(i) \leqslant z = \|\xi\|_k$ then $F[\mathbb{I}(i)] = (\phi_k^{\|\xi\|_{k'}})^{-1}[\mathbb{I}(i)] = \phi_{k'}^{\|\xi\|_l} \circ \pi_\xi^{k,l}[\mathbb{I}(i)]$. By virtue of the hypotheses, $\pi_\xi^{k,l}[\mathbb{I}]$ is an interval in m_l which is contained in $(\|\xi\|_l)_k$. Hence, such interval is also included in $(\|\xi\|_l)_{k'}$. Since, $\phi_{k'}^{\|\xi\|_l}$ is the increasing bijection from $(\|\xi\|_l)_{k'}$ to $\|\xi\|_{k'}$, it follows that $\phi_{k'}^{\|\xi\|_l}[\pi_\xi^{k,l}[\mathbb{I}]]$ is an interval in $\|\xi\|_{k'} + 1$. From this it is straightforward that $F[\mathbb{I}(i)]$ is an interval in $m_{k'}$. This finishes the claim. $\qquad \square$

Since $(\mathbb{1}, z) \in T$ and $(F, \mathbb{1}) \in \mathcal{Q}$ then $(\mathbb{1}_F, F(z)) \in T'$. We will show that such pair testifies that $T' \bigstar (\alpha, \xi, k', l)$. This will be achieved by proving the three following claims.

<u>Claim 2</u>: $\|\alpha\|_l \leqslant \|\xi\|_l$ and $\|\xi\|_{k'} = F(z)$.

Proof of claim. We know that $z = \|\xi\|_k$, and it has already been proved that $F(\|\xi\|_k) = \|\xi\|_{k'}$. As $\alpha \in (\beta)_{k'}$ then $\|\alpha\|_l \leqslant \|\beta\|_l$. But we know by the hypotheses that $\|\beta\|_l \leqslant \|\xi\|_l$. So we are done

$\square$

<u>Claim 3</u>: $\pi_\xi^{k',l} \circ \mathbb{1}_F \in Bl(0, m_l)$.

Proof of claim. Let $i < |\mathbb{1}_F|$. We want to show that $\pi_\xi^{k',l}[\mathbb{1}_F(i)]$ is an interval in m_l. First we deal with the case where $i = 0$. Here, $\mathbb{1}_F(i) = (\|\alpha\|_{k'}, \|\beta\|_{k'}]$. We will first argue that, in this case, $\pi_\xi^{k',l}[\mathbb{1}_F(i)] = \pi_\beta^{k',l}[\mathbb{1}_F(i)]$. Indeed, since $\|\beta\|_l \in (\|\xi\|_l)_{k'}$ then $\phi_{k'}^{\|\xi\|_l}|_{\|\beta\|_l + 1} = \phi_{k'}^{\|\beta\|_l}$. In particular, this means that $(\phi_{k'}^{\|\xi\|_l})^{-1}[a] = (\phi_{k'}^{\|\beta\|_l})^{-1}[a]$ for each $a \leqslant \|\beta\|_{k'}$. In this way, by using Lemma 5.0.24, we conclude that

$$\begin{aligned} \pi_\xi^{k',l}[\mathbb{1}_F(i)] &= \phi_l^\xi \circ (\phi_{k'}^\xi)^{-1}[\mathbb{1}_F(i)] \\ &= \phi_l^\xi \circ ((\phi_l^\xi)^{-1} \circ (\phi_{k'}^{\|\xi\|_l})^{-1})[\mathbb{1}_F(i)] \\ &= (\phi_{k'}^{\|\beta\|_l})^{-1}[\mathbb{1}_F(i)] = (\phi_{k'}^{\|\beta\|_l})^{-1}[\mathbb{1}_F(i)] \\ &= (\phi_l^\beta \circ ((\phi_l^\beta)^{-1}) \circ (\phi_{k'}^{\|\beta\|_l})^{-1}[\mathbb{1}_F(i)] \\ &= \phi_l^\beta \circ (\phi_{k'}^\beta)^{-1}[\mathbb{1}_F(i)] = \pi_\beta^{k',l}[\mathbb{1}_F(i)] \end{aligned}$$

Observe that $(\phi_{k'}^\beta)^{-1}[\mathbb{1}(i)] = (\beta)_{k'}[(\|\alpha\|_{k'}, \|\beta\|_{k'}]]$ is an interval in $(\beta)_l$, according to the point (3) in the hypotheses of this lemma. Since ϕ_l^β is the increasing bijection from $(\beta)_l$ to $\|\beta\|_l + 1 \sqsubseteq m_l$, then $\pi_\beta^{k',l}[\mathbb{1}(i)]$ is an interval in m_l. This finishes the first case. Now we deal with the case where $i > 0$. Here, $\mathbb{1}_F(i) = F[\mathbb{1}(i-1)]$. Thus,

$$\begin{aligned} \pi_\xi^{k',l}[\mathbb{1}_F(i)] &= \phi_l^\xi \circ (\phi_{k'}^\xi)^{-1}[F(i)] \\ &= \phi_l^\xi \circ (\phi_{k'}^\xi)^{-1}[(\phi_k^{\|\xi\|_{k'}})^{-1}[\mathbb{1}(i-1)]] \\ &= \phi_l^\xi \circ ((\phi_{k'}^\xi)^{-1} \circ (\phi_k^{\|\xi\|_{k'}})^{-1})[\mathbb{1}(i-1)] \\ &= \phi_l^\xi \circ (\phi_k^\xi)^{-1}[\mathbb{1}(i-1)] = \pi_\xi^{k,l}[\mathbb{1}(i-1)] \end{aligned}$$

Due to the hypotheses, $\pi_\xi^{k,l}[\mathbb{1}(i-1)]$ is already an interval in m_l. Consequently, this case is over.

$\square$

<u>Claim 4</u>: $\|\alpha\|_l \in (\|\xi\|_l)_{k'}$.

Proof of claim. Recall that $\alpha \in (\beta)_k$. Therefore, $\|\alpha\|_l \in (\|\beta\|_l)_k$ due to the point (b) of Proposition 5.0.23. As $\|\beta\|_l \in (\|\xi\|_l)_{k'}$, we are done.

$\square$

$\square$

$(b) \Rightarrow (a)$. Let $(\mathbb{I}, z) \in T$ and $F \in \mathcal{F}_k(m_{k'})$ with $(F, \mathbb{I}) \in \mathcal{Q}$ be such that $\|\alpha\|_{k'} \leqslant \|\xi\|_{k'} = F(z)$, $\pi_\xi^{k',l} \circ \mathbb{I}_F \in Bl(0, m_l)$ and $\|\alpha\|_l \in (\|\xi\|_l)_{k'}$. We will show in the next three claims that $(\mathbb{I}, z)$ testifies that $T \bigstar (\beta, \xi, k', l)$.

<u>Claim 5</u>: $\|\beta\|_l \leqslant \|\xi\|_l$ and $\|\xi\|_k = z$.

Proof of claim. Due to the hypotheses of the lemma, we already know that $\|\beta\|_l \leqslant \|\xi\|_l$. Therefore it suffices to prove that $\|\xi\|_k = z$. Indeed, as $F \in \mathcal{F}_k(m_{k'})$ then $\|F(z)\|_k = z$. Moreover, we know that $F(z) = \|\xi\|_{k'}$. Thus, $z = \|F(z)\|_k = \| \|\xi\|_{k'} \|_k = \|\xi\|_k$ by virtue of the point (a) in Proposition 5.0.23. $\square$

<u>Claim 6</u>: $\pi_\xi^{k,l} \circ \mathbb{I} \in Bl(0, m_l)$.

Proof of claim. Let $i < |\mathbb{I}|$. We need to prove that $\pi_\xi^{k,l}[\mathbb{I}(i)]$ is an interval in m_l. First note that $\|\xi\|_{k'} \in F$, so as in the proof of the other direction of the lemma, $F \cap (\xi + 1) = (\|\xi\|_{k'})_k$. In this way, $F(a) = (\phi_k^{\|\xi\|_{k'}})^{-1}(a)$ for each $a \leqslant \|\xi\|_k$. From this fact, we get the following chain of equalities:

$$\pi_\xi^{k,l}[\mathbb{I}(i)] = \phi_l^\xi \circ (\phi_k^\xi)^{-1}[\mathbb{I}(i)] = \phi_l^\xi \circ ((\phi_{k'}^\xi)^{-1} \circ (\phi_k^{\|\xi\|_{k'}})^{-1})[\mathbb{I}(i)]$$
$$= \phi_l^\xi \circ \phi_{k'}^\xi[F[\mathbb{I}(i)]]$$
$$= \pi_\xi^{k',l}[\mathbb{I}_F(i+1)]$$

As $\pi_\xi^{k',l} \circ \mathbb{I} \in Bl(0, m_l)$, we are done. $\square$

<u>Claim 7</u>: $\|\beta\|_l \in (\|\xi\|_l)_k$.

Proof of claim. On one hand, $\|\beta\|_{k'} \in F$ since $(F, \mathbb{I}) \in \mathcal{Q}$. On the other hand, we already know that $\|\xi\|_k \in F$. Therefore, $\rho(\|\beta\|_{k'}, \|\xi\|_{k'}) \leqslant k$. Now, according to the hypotheses $\rho(\|\xi\|_l, \|\beta\|_l) \leqslant k'$. Thus,

$$\rho(\|\beta\|_l, \|\xi\|_l) = \rho(\| \|\beta\|_l \|_{k'}, \| \|\xi\|_l \|_{k'}) = \rho(\|\beta\|_{k'}, \|\xi\|_{k'})$$

by virtue of the points (a) and (b) of Proposition 5.0.23. In this way, $\rho(\|\beta\|_l, \|\xi\|_l) \leqslant k$. $\square$

$\square$

$\blacksquare$

Corollary 5.2.20. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ. Also, let $\beta \in \gamma$, $T \in Good_k(\beta)$, $k' > k$ and $\alpha \in (\beta)_{k'}^-$. Suppose that $(\beta)_{k'} \backslash (\alpha + 1)$ is an interval in $(\beta)_{l-1}$. Furthermore, assume that $l > k'$ and $\delta \in Lim \cap \alpha$ are such that $|(\alpha)_{l-1} \cap \delta| = r_l$. If $C \in FIN(\gamma)$ is such that $(C, \delta) \checkmark (k, l, \beta, \delta, T)$ then $(C, \delta) \checkmark (k', l, \beta, \delta, T')$ where $T' = Transf_{k'}(\alpha, \beta, T)$. In particular, $j(\delta, T, \beta, k, l) \leqslant j(\delta, T', \alpha, k', l)$.*

Proof. Let $C \in \mathrm{FIN}(\gamma)$ be such that $(C, \delta) \checkmark (k, l, \beta, \delta, T')$. Since $\alpha \in (\beta)_{k'}$ and $l > k'$ then $(\beta)_l \cap (\alpha + 1) = (\alpha)_l$. By the hypotheses, $\alpha \geqslant \delta$. This means that $(C(0))_l[r_l] = (\beta)_l \cap \delta = (\alpha)_l \cap \delta$. The only thing left to prove is that $T' \bigstar (\alpha, C(0), k', l-1)$. For this, it suffices to show that the hypotheses of Lemma 5.2.19 hold for $\xi = C(0)$. Indeed, since $T \bigstar (\beta, \xi, k, l-1)$ then $\|\beta\|_{l-1} \in (\|\xi\|_{l-1})_k$. This implies both that $\|\beta\|_{l-1} \leqslant \|\xi\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|\xi\|_{l-1})_{k'}$. According to the assumptions $(\beta)_{k'}[(\|\alpha\|_{k'}, \|\beta\|_{k'}]]$ is an interval in $(\beta)_{l-1}$. Thus, we are done.

∎

5.2.1 Forcing the property IH_2

In this subsection we aim to prove that if γ is a countable ordinal and $\mathcal{F}$ satisfies IH_1 and IH_2, then we can extend $\mathcal{F}$ to a construction scheme over $\gamma + \omega$ which also satisfies IH_1 and IH_2. For this, we will make use of the forcing $\mathbb{P}(\mathcal{F})$. In Section 5.1 we already proved that there is a countable family of dense sets over $\mathbb{P}(\mathcal{F})$ so that if G is a filter intersecting each of those sets, then $\mathcal{F}^{\mathcal{G}}$ is construction scheme over $\gamma + \omega$ which also satisfies IH_1. Here we want to show the same result but for the Property IH_2. However, due to the large amount of variables appearing in the Definition 5.2.13, the explicit formula for each of those dense sets is quite messy. For that reason, we will instead show that $\mathbb{P}(\mathcal{F})$ forces $\mathcal{F}^{Gen}$ to satisfy IH_2. Before doing this, it is worth pointing out some last remarks regarding the forcing $\mathbb{P}(\mathcal{F})$.

Remark 5.2.21. By virtue of the the Lemmas 5.1.17,and 5.1.19, we have that

$$\mathbb{P}(\mathcal{F}) \Vdash \text{`` } \mathcal{F}^{Gen} \text{ is construction scheme over } \gamma + \omega \text{ which satisfies } IH_1 \text{''}.$$

Remark 5.2.22. $\mathbb{P}(\mathcal{F}) \Vdash$ `` $\mathcal{F} \subseteq \mathcal{F}^{Gen}$'' due to the Lemma 5.1.18. This means that $\rho_{F^{Gen}}|_{\gamma^2}$ is forced to be equal to $\rho_{\mathcal{F}}$. Therefore, there is no risk of confusion by referring to both of these two ordinal metrics simply as ρ.

Remark 5.2.23. If $p \in \mathbb{P}_k(\mathcal{F})$ for some $k \in \omega$, then $p \vDash$ `` $p \in \mathcal{F}_k^{Gen}$ ''. In particular, $\mathcal{F}(p)$ is forced to be equal to $\mathcal{F}^{Gen}|_p$. In other words, $\rho|_{p^2} = \rho_{\mathcal{F}(p)}$. So, again, there is no risk of confusion on referring to $\rho_{\mathcal{F}(p)}$, simply as ρ.

Remark 5.2.24. Suppose that $p \in \mathbb{P}_l(\mathcal{F})$, $\beta \in p$ and $k \leqslant l$. Then for each $H \in \mathcal{F}_j(p)$ it happens that $p \Vdash$ `` $H \cap (\beta + 1) = (\beta)_k$ ''.[4] That is, p decides the value of $(\beta)_k$. Furthermore, this value can be calculated using $\mathcal{F}(p)$. From this, it follows that p also knows who is $\|\beta\|_k$, $\Xi_\beta(k)$, $Bl_k(\beta)$ $(= Bl(\|\beta\|_k, m_k))$ and $Good_k(\beta)$. Therefore, there should not be any confusion while working explicitly with this sets instead of with their names, of course, as long as the conditions at the beginning of this remark hold.

Lemma 5.2.25. *Let $q \in \mathbb{P}(\mathcal{F})$. Then there is $p \leqslant q$ so that $|p \cap \gamma| = r_{k_p+1}$.*

Proof. From Lemma 5.1.17, it is easy to get $q' \leqslant q$ so that $k_{q'} > k_q$. Now, according to Lemma 5.1.19, there is $p' \in \mathbb{P}(\mathcal{F})$ so that $p' \leqslant q'$ and $p' \cap \gamma = R(p')$. Put $p = p'_0$. Then $p \in \mathbb{P}(\mathcal{F})$ and $p \leqslant q$. Furthermore, $|p \cap \gamma| = |R(p')| = r_{k_{p'}} = r_{k_p+1}$.

∎

---

[4] Here, by $(\beta)_k$ we mean k-closure of β calculated inside $\gamma + \omega$ with respect to $\rho_{\mathcal{F}^{Gen}}$.

Theorem 5.2.26. *Suppose that γ is countable and $\mathcal{F}$ satisfies IH_1 and IH_2. Then*

$$\mathbb{P}(\mathcal{F}) \Vdash \text{ `` } \mathcal{F}^{Gen} \text{ satisfies } IH_2 \text{ ''}.$$

Proof. According to the Definition 5.2.13, our goal is to prove that

$$\mathbb{P}(\mathcal{F}) \Vdash \text{ `` } IH_2(\delta, \mathcal{F}^{Gen}) \text{ holds ''}$$

for any $\delta \in \mathrm{Lim} \cap (\gamma + \omega)$. If $\mathcal{F}$ satisfies that there are infinitely many $l \in \omega$ for which there is $C \in \mathrm{FIN}(D_\delta)$ which is fully captured at level l, it is easy to see that $\mathbb{P}(\mathcal{F}) \vDash \text{ `` } IH_2^A(\gamma, \mathcal{F}^{Gen}) \text{ holds ''}$. This is because $\mathbb{P}(\mathcal{F}) \Vdash \text{ `` } \mathcal{F} \subseteq \mathcal{F}^{Gen} \text{ ''}$. Therefore, the only interesting case happens when:

With respect to $\mathcal{F}$, there is $k'' \in \omega$ such that for each $l \geqslant k''$ there is no $C \in \mathrm{FIN}(D_\gamma)$ which is fully captured at level l.

From now on, let us assume that this case holds.

<u>Claim</u>: $\mathbb{P}(\mathcal{F}) \Vdash \text{ `` } IH_2^B(\delta, \mathcal{F}^{Gen}) \text{ holds ''}$.

Proof of claim. Let $q \in \mathbb{P}(\mathcal{F})$, $\beta \in [\delta, \gamma + \omega)$, $2 \leqslant k \in \omega$ and a name $\dot{T}$ for an element of $Good_k(\beta)$. Because of Lemma 5.1.17 , we may assume without loss of generality that $\beta \in q$ and $q \in \mathbb{P}_{k'}(\mathcal{F})$ for some $k' > k, k''$. According to the Remark 5.2.24, $Good_k(\beta)$ is fully determined by q. In particular, this means that there is T such that $q \Vdash \text{ `` } T = \dot{T} \text{ ''}$.

We need to show that there is $l > k'$ and $p \leqslant q$ with the following properties:

- $p \Vdash \text{ `` } |(\beta)_{l-1} \cap \delta| = r_l \text{ ''}$.

- There is $j \in \omega$ so that $p \Vdash \text{ `` } j = j(\delta, T, \beta, k, l) \text{ ''}$.

- $p \Vdash \text{ `` } \Xi_\beta(l) = j \text{ ''}$.

- If $j > 0$ then there is $C \in [\gamma + \omega]^j$ so that $p \Vdash \text{ `` } (C, \delta)\checkmark(T, \beta, k, l) \text{ and } C \subseteq (\beta)_l \text{ ''}$.

We will divide the rest of the proof into two cases.

<u>Case 1</u>: $\delta = \gamma$.

Proof of case. By virtue of the Lemma 5.2.25, there is $q' \in \mathbb{P}(\mathcal{F})$ so that $q' \leqslant q$ and $|q' \cap \gamma| = r_{k_q+1}$. Let $l = k_q + 1$. Note that the objects needed to calculate $j(\gamma, T, \beta, k, l)$ are $\mathcal{F}^{Gen}|_\gamma$, T, $(\beta)_{l-1}$, k an l. All of this objects are already decided by q'. Therefore, there is $j \in \omega$ so that $q' \Vdash \text{ `` } j = j(\gamma, T, \beta, k, l) \text{ ''}$.

<u>Subcase 1.1</u>: If $j = 0$.

Proof of subcase. Consider $F \in \mathcal{F}_l$ so that $q' \cap \gamma \sqsubseteq F$. Then $R(F) = q' \cap \gamma$. Let $\alpha = \min(F_0 \backslash R(F))$ and $p = Cut_\alpha(F)$. It is straightforward that $R(p) = q' \cap \gamma$ and $q' = p_0$. This means that $p \leqslant q$. Moreover, as $\beta \in q \backslash \gamma$ then $p \Vdash \text{ `` } |(\beta)_{l-1} \cap \delta| = r_l \text{ ''}$ and $p \Vdash \text{ `` } \Xi_\beta(l) = 0 = j \text{ ''}$. In this case, there is nothing more to prove. □

<u>Subcase 1.2</u>: If $j > 0$.

Proof of subcase. By definition of j, there is $C \in [\gamma + \omega]^j$ so that $(C, \gamma) \checkmark (T, \beta, k, l)$. According the point (1) of Definition 5.2.12, $C \subseteq D_\gamma$ and C is captured at level l. Let $F \in \mathcal{F}_l$ be so that $C \subseteq F$. Then, by the point (3) of Definition 5.2.12, it follows that $R(F) = (\beta)_{l-1} \cap \gamma = q' \cap \gamma$. In this case, let $\alpha = \min(F_j \backslash R(F))$ and $p = Cut_\alpha(F)$. As in the previous case, we have $p_j = q'$ and $R(p) = q' \cap \gamma$. Thus, $q' \leqslant p_j$. Again, as $\beta \in q \backslash \gamma$ then $p \Vdash$ " $|(\beta)_{l-1} \cap \delta| = r_l$ " and $p \Vdash$ " $\Xi_\beta(l) = j$ ". In order to finish, note that $p \cap \gamma = \bigcup_{i<j} F_i$. In particular, $C \subseteq p \cap \gamma \subseteq (\beta + 1) \cap p = (\beta)_l$. That is,

$$p \Vdash \text{ `` } (C, \gamma) \checkmark (T, \beta, k, l) \text{ and } C \subseteq (\beta)_l \text{ ''.}$$

$\square$

$\square$

<u>Case 2</u>: If $\delta < \gamma$.

Proof of case. If $\beta < \gamma$ there is nothing to do. This is because $\mathcal{F}$ is forced to be contained in $\mathcal{F}^{Gen}$ and $\mathcal{F}$ already satisfies IH_2. Therefore, we may assume that $\beta \geqslant \gamma$. Appealing to Lemma 5.1.17, we may assume without loss of generality that $q \cap [\delta, \gamma) \neq \emptyset$. Let $\alpha = \alpha_q = \max(q \cap \gamma) \geqslant \delta$. The plan is apply the property IH_2 to α (inside $\mathcal{F}$). For this purpose, let $\mathcal{Q} = \mathcal{Q}_{k'}(\beta, T)$ and $T' = \mathrm{Transf}_{k'}(T, \alpha, \beta)$.

By our assumptions $IH_2^A(\delta, \mathcal{F})$ does not hold. Therefore, $IH_2^B(\delta, \mathcal{F})$ holds. This means that there is $l \geqslant k'$ for which $j' = j(\delta, T', \alpha, k', l) < n_l$ and such that:

(I) $|(\alpha)_{l-1} \cap \delta| = r_l$,

(II) $\Xi_\alpha(l) = j$,

(III) If $j > 0$ then there is $C \in [\gamma]^j$ for which $(C, \delta) \checkmark (T', \alpha, k', l)$ and $C \subseteq (\alpha)_l$.

Before we continue, we remark that:

- $q \Vdash$ " $\alpha \in \beta \cap q = (\beta)_{k'}^-$ ".

- $q \Vdash$ " $(\beta)_{k'} \backslash (\alpha + 1) = [\gamma, \beta + 1)$ ". Since $[\gamma, \beta + 1)$ is already an interval in $\gamma + \omega$, it follows that $q \Vdash$ " $(\beta)_{k'}$ is an interval in $(\beta)_{l-1}$".

- $q \Vdash$ " $(\beta)_{l-1}^- \cap \alpha = (\alpha)_{l-1}$ " because $l > k'$. In particular, this means that $q \Vdash$ " $|(\beta)_{l-1} \cap \delta| = |(\alpha)_{l-1} \cap \delta| = r_l$ ".

- Since $j' \geqslant 0$ and $\Xi_\alpha(l) = j'$ then $q \Vdash$ " $\Xi_\beta(l) = j'$ " by means of the Lemma 2.3.9.

Due to the first two points, the hypotheses of Corollary 5.2.20 are fulfilled. In particular, this means that

$$q \Vdash \text{ `` } j(\delta, T, \beta, k, l) \leqslant j' \text{ ''.}$$

We will divide the rest of the proof into two subcases.

<u>Subcase 2.1</u>: If $j' = 0$.

Proof of subclaim. As $j(\delta, T, \beta, k, l)$ is always non-negative then

$$q \Vdash \text{``} j(\delta, T, \beta, k, l) = 0 \text{''}.$$

Because of this and by the last two points of the previous remark, $p = q$ forces everything that we want. $\qquad\square$

<u>Subcase 2.2</u>: If $j' > 0$.

Proof of subcase. Let $F \in \mathcal{F}_l$ be such that $\alpha \in F$. Since $j > 0$, there is $C \in [\gamma]^j$ for which $(C, \delta)\checkmark(T', \alpha, k', l)$ and $C \subseteq (\alpha)_l$. Note that $q \vDash \text{``}(\alpha)_l \subseteq (\beta)_l\text{''}$. In this way, $q \Vdash \text{``} C \subseteq (\beta)_l \text{''}$. Our next task will be to extend q to a condition forcing that $T\bigstar(\beta, C(0), k, l-1)$. By appealing to Lemma 5.2.19, it suffices to find $p \leqslant q$ which forces that $\|\beta\|_{l-1} \leqslant \|C(0)\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|\xi\|_{l-1})_{k'}$.

As $(C, \delta)\checkmark(T', \alpha, k', l)$ then $T'\bigstar(\alpha, C(0), k', l-1)$. Let $\xi \in F_j$ be such that $\|\xi\|_{l-1} = \|C(0)\|_{l-1}$. By Remark 5.2.10, $T'\bigstar(\alpha, \xi, k', l-1)$. Thus, there are $(\mathbb{I}, z) \in T$ and $(G, \mathcal{I}) \in \mathcal{Q}$ so that $(\mathbb{I}_G, G(z))$ testifies the previous guessing relation. That is:

(a) $\|\alpha\|_{l-1} \leqslant \|\xi\|_{l-1}$ and $\|\xi\|_{k'} = G(z)$. Since both ξ and α are in F_j then $\rho(\alpha, \xi) \leqslant l-1$. Thus, this point implies that $\alpha \leqslant \xi$.

(b) $(\xi)_{k'} \circ \mathbb{I}_G \in Bl(0, (\xi)_{l-1})$.

(c) $(\xi)_{l-1}(\|\alpha\|_{l-1}) \in (\xi)_{k'}$. As $\alpha, \xi \in F_j$ then $(\xi)_{l-1}(\|\alpha\|_{l-1}) = F_j(\|\alpha\|_{l-1}) = \alpha$. Therefore, this point implies that $\alpha \in (\xi)_{k'}$.

According to the point (b), $(\xi)_{k'}[\mathbb{I}_G(0)] = (\xi)_{k'}[(\|\alpha\|_{k'}, \|\beta\|_{k'})]$ is an interval in $(\xi)_{l-1}$. Moreover, by the point (a), this interval is contained in $F_j \backslash R(F)$. Therefore, it is also an interval in F. Let $\alpha' = (\xi)_{k'}[\|\alpha\|_{k'} + 1]$. In other words, α' is the successor of α inside $(\xi)_{k'}$. Because of this, it follows that $(\alpha')_{k'}^- = (\alpha)_{k'} = q \cap \gamma$. In this way, $p = Cut_{\alpha'}(F) \leqslant q$ due to the Lemma 5.1.10.

<u>Subclaim 1</u>: $p \Vdash \text{``} \|\beta\|_{l-1} \leqslant \|C(0)\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|C(0)\|_{l-1})_{k'} \text{''}$.

Proof of subclaim. Let $\phi : F \longrightarrow p$ be the increasing bijection. By the definition of the Cut function, it follows that $\phi(\alpha') = \gamma$. Now, since $(\xi)_{k'}[\mathbb{I}_F(0)]$ is an interval in F, then $\phi[(\xi)_{k'}[\mathbb{I}_G(0)]] = (\phi(\xi))_{k'}[\mathbb{I}_G(0)]$ is an interval in p. Furthermore, the first point of this interval is γ. In this way, as $p\backslash\gamma$ is an initial segment of $[\gamma, \gamma + \omega)$, we conclude that $\phi((\xi)'_k(\|\beta\|_{k'}))$ (the last point of the interval $\phi[(\xi)_{k'}[\mathbb{I}_G(0)]]$) is equal to

$$\gamma + (\|\beta\|_{k'} - \|\alpha\|_{k'}) = \beta.^5$$

As direct consequence of this equality, p forces that $\beta \leqslant \phi(\xi)$ and $\rho(\beta, \phi(\xi)) \leqslant k'$. It then follows that $\|\beta\|_{l-1} \leqslant \|\phi(\xi)\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|\phi(\xi)\|_{l-1})_{k'}$. Since $\|\phi(\xi)\|_{l-1} = \|\xi\|_{l-1} = \|C(0)\|_{l-1}$, we are done. $\qquad\square$

---

[5]All the definitions related to IH_2 were originally formulated to achieve this single step of the proof.

By virtue of the previous subclaim, $p \Vdash$ " $T \bigstar (\beta, C(0), k, l-1)$ ". It then follows that $p \Vdash$ " $(C, \delta) \checkmark (\delta, T, \beta, k, l)$ ". As $|C| = j'$, we get that

$$p \Vdash \text{ `` } j' \leqslant j(\delta, T, \beta, k, l) \text{ ''.}$$

Hence, these two numbers are equal. In conclusion, we have shown that there are $p \leqslant q$ and $j \in \omega$ (namely, j'), so that:

- $p \Vdash$ " $|(\beta)_{l-1} \cap \delta| = r_l$ " and $p \Vdash$ " $\Xi_\beta(l) = j$ ".[6]

- There is $C \in [\gamma + \omega]^j$ so that $p \Vdash$ " $(C, \delta) \checkmark (T, \beta, k, l)$ and $C \subseteq (\beta)_l$".

Thus, the proof of this subcase is over. $\square$

$\square$

$\square$

$\blacksquare$

It is not hard to see that the property forced in the previous theorem can be coded by countable many dense subsets of $\mathbb{P}(\mathcal{F})$. In this way, we have the following corollary.

Corollary 5.2.27. *Suppose that γ is a limit ordinal and $\mathcal{F}$ satisfies IH_1 and IH_2. There is a countable family of dense sets in $\mathbb{P}(\mathcal{F})$ so that if $\mathcal{G}$ is a filter intersecting all of them, then $\mathcal{F}^{\mathcal{G}}$ is a construction scheme over $\gamma + \omega$ which contain $\mathcal{F}$ and satisfies both IH_1 and IH_2.*

Now we have proved all the necessary results to show that there is a fully capturing construction scheme over ω_1 of type τ. Namely, Propositions 5.1.12 and 5.2.14 , Lemmas 5.1.13 and 5.2.15, and Corollary 5.2.27. From them we conclude that there is a sequence $\langle \mathcal{F}^\delta \rangle_{\delta \in \mathrm{Lim}}$ so that for all $\delta < \gamma \in \mathrm{Lim}$, the following properties hold:

- $\mathcal{F}_\delta$ is a construction scheme over δ satisfying IH_1 and IH_2,

- $\mathcal{F}_\delta \subseteq \mathcal{F}_\gamma$.

It follows that $\mathcal{F} = \bigcup_{\delta \in \mathrm{Lim}} \mathcal{F}^\delta$ is a fully capturing construction scheme. This is due to the Theorem 5.2.16.

Theorem 5.2.28. *The $\Diamond$-principle implies FCA.*

Now fix $\mathcal{P}$ an aribtrary partition of ω compatible with τ. Consider the following variations of the the properties IH_1 and IH_2.

Definition 5.2.29 (The $IH_1(\mathcal{P})$ property)**.** We say that $\mathcal{F}$ satisfies $IH_1(\mathcal{P})$ if for all $A \in \mathrm{FIN}(\gamma)$, $P \in \mathcal{P}$ and $\alpha \in \gamma$, there is $F \in \mathcal{F}$ with $\rho^F \in P$ such that $IH_1(\alpha, A, F)$ holds.

---

[6]This was already forced by q.

Definition 5.2.30 (The $IH_2(\mathcal{P})$ property)**.** Given $\delta \in \mathrm{Lim} \cap \gamma$ and $P \in \mathcal{P}$, we say that $IH_2(\delta, \mathcal{F})$ holds if one of the two following mutually excluding conditions occurs:

(A) There are infinitely many $l \in P$ for which there is $C \in \mathrm{FIN}(D_\delta)$ which is fully captured at level l. In this case, we will say that $IH_2^A(P, \delta, \mathcal{F})$ holds.

(B) For all $\beta \in [\delta, \gamma)$, $2 \leqslant k \in \omega$ and $T \in Good_k(\beta)$ there are infinitely many $k \leqslant l \in P$ for which $j = j(\delta, T, \beta, k, l) < n_l$ and such that:

    (B.1) $|(\beta)_{l-1} \cap \delta| = r_l$,

    (B.2) $\Xi_\beta(l) = j$,

    (B.3) If $j > 0$ then there is $C \in [\delta]^j$ for which $(C, \delta)\checkmark(T, \beta, k, l)$ and $C \subseteq (\beta)_l$.

    In this case we will say that $IH_2^B(P, \delta, \mathcal{F})$ holds.

Finally, we say that $\mathcal{F}$ satisfies $IH_2(\mathcal{P})$ if $IH_2(\delta, \mathcal{F})$ holds for any $\delta \in \mathrm{Lim} \cap \gamma$ and each $P \in \mathcal{P}$.

All the results stated in this section are true when IH_1 is changed by $IH_1(\mathcal{P})$ and IH_2 is changed by $IH_2(\mathcal{P})$. Moreover, the proofs are completely similar. In this way, we deduce theorem 2.4.19.

Theorem 5.2.31. *The $\Diamond$-principle implies FCA(part).*

5.3   P-ideal Dichotomy and capturing axioms

Throughout this work, we have seen a handful of applications of construction schemes. Many of the objects that we have constructed using capturing construction schemes already contradict to some degree the P-ideal dichotomy. For example, since the Suslin Hypothesis follows from PID (see [4] and [123]), this axiom is incompatible with FCA. In addition, since PID implies that any gap is indestructible, CA_3 also contradicts PID. However, note that none of the applications that we have of CA_2 is enough to contradict the P-ideal dichotomy. For now, the best we have is that $PID + \min(\mathfrak{b}, cof(\mathcal{F}_\sigma)) > \omega_1$ is incompatible with the existence of a 2-capturing construction scheme. This is because such assertion is equivalent to the nonexistence of a sixth Tukey type (see [85]). The purpose of this section is to prove that PID is in fact incompatible with CA_2.

For the remainder of this section, we will assume that PID holds and that $\mathcal{F}$ is a construction scheme. Our plan is to define a P-ideal $\mathcal{I}$ from $\mathcal{F}$ that contradicts PID whenever $\mathcal{F}$ is 2-capturing. This ideal $\mathcal{I}$ will be defined as family of countable sets of the orthogonal ideal of some other ideal $\mathcal{H}$. When applying the second alternative of PID to the ideal $\mathcal{I}$, we will need to calculate who is $\mathcal{I}^\perp$. In general, this may be a difficult task. However, the following concept will facilitate these calculations.

Definition 5.3.1. Let $\mathcal{H}$ be an ideal of countable sets in ω_1. We will say that $\mathcal{H}$ is *Frechet in* $[\omega_1]^\omega$ if for any $A \in \mathcal{H}^+$, there is an infinite $B \subseteq A$ such that $B \in \mathcal{H}^\perp$.

Proposition 5.3.2. *Let $\mathcal{H}$ be an ideal of countable sets in ω_1 and assume that $\mathcal{H}$ is Frechet in $[\omega_1]^{\omega_1}$. Then $(\mathcal{H}^\perp \cap [\omega_1]^{\leqslant\omega})^\perp \cap [\omega_1]^\omega = \mathcal{H} \cap [\omega_1]^\omega$.*

Proof. We will only show that the inclusion from right to left holds. For this purpose, let $A \in (\mathcal{H}^\perp \cap [\omega_1]^{\leqslant\omega})^\perp \cap [\omega_1]^\omega$. Assume towards a contradiciton that $A \notin \mathcal{H}$. Then $A \in \mathcal{H}^+ \cap [\omega_1]^\omega$. Since $\mathcal{H}$ is Frechet, there is an infinite $B \subseteq A$ such that $B \in \mathcal{H}^\perp$. In particular, $A \cap B$ is infinite, and this contradicts the fact that $A \in (\mathcal{H}^\perp)^\perp$. Thus, the proof is over.

∎

We now define the ideal $\mathcal{I}$ which will be used in the proof of Theorem 5.3.11.

Definition 5.3.3. Given $n \in \omega$ and $\alpha \in \omega_1$, we define

$$H_n(\alpha) = \{\xi < \alpha \,:\, \forall m > n\,(\,\Xi_\alpha(m) = -1 \ or \ \Xi_\xi(m) \leqslant \Xi_\alpha(m)\,)\}.$$

Additionally, we define the ideal generated by the family $\{H_n(\alpha) \,:\, n \in \omega \ and \ \alpha \in \omega_1\}$ as $\mathcal{H}$. Lastly, we define $\mathcal{I} = \mathcal{H}^\perp \cap [\omega_1]^{\leqslant\omega}$.

Remark 5.3.4. $[\omega_1]^{<\omega}$ is included in both $\mathcal{H}$ and $\mathcal{I}$.

The following lemma is easy.

Lemma 5.3.5. *Let $\alpha, \beta, \gamma < \omega_1$ and $m, n \in \omega$. Then:*

(1) *If $m < n$, then $H_m(\alpha) \subseteq H_n(\alpha)$.*

(2) *If $\beta < \gamma$ and $n > \rho(\beta, \gamma)$, then $H_n(\gamma) \cap \beta \subseteq H_n(\beta)$.*

Proof. The point (1) follows directly from the definition. In order to prove (2) we will show that $\omega_1 \backslash H_n(\beta) \subseteq \omega_1 \backslash (H_n(\gamma) \cap \beta)$. Let $\xi \in \omega_1 \backslash H_n(\beta)$. If $\xi \geqslant \beta$ we are done, so let us assume that $\xi < \beta$. Then there is $m > n$ for which $\Xi_\xi(m) > \Xi_\beta(m) \geqslant 0$. Note that $m > \rho(\beta, \gamma)$. Thus, by the point (c) of Lemma 2.3.9 we conclude that $\Xi_\beta(m) = \Xi_\gamma(m)$. In this way, $\xi \notin H_n(\gamma)$. This finishes the proof.

∎

Remark 5.3.6. By the point (1) of the previous lemma we have that for any $A \in \mathcal{H}$ there are $n \in \omega$ and $\alpha_1, \ldots, \alpha_n \in \omega_1$ for which $A \subseteq \bigcup_{i < n} H_n(\alpha_i)$.

Definition 5.3.7. Given $\beta \in \omega_1$ we define $\mathcal{H}|_\beta$ as the ideal generated by the family $\{H_n(\alpha) \,:\, \alpha \leqslant \beta \ and \ n \in \omega\}$.

Lemma 5.3.8. *Let $\beta < \omega_1$ and $A \subseteq \beta$.*

(a) *If $A \in (\mathcal{H}|_\beta)^\perp$, then $A \in \mathcal{H}^\perp$. Furthermore, since A is countable then $A \in \mathcal{I}$.*

(b) *If $A \in (\mathcal{H}|_\beta)^+$, then $A \in \mathcal{H}^+$.*

Proof. *Proof of* (a). It is enough to show that $A \cap H_n(\gamma) =^* \emptyset$ for any $\beta \leqslant \gamma \in \omega_1$ and each $n \in \omega$. Furthermore, we may assume that $n > \rho(\beta, \gamma)$ due to the point (1) of Lemma 5.3.5. Indeed, by virtue of the point (2) of the same lemma, we have that

$$A \cap H_n(\gamma) = A \cap \beta \cap H_n(\gamma) \subseteq A \cap H_n(\beta) =^* \emptyset.$$

$\square$

Proof of (b). Suppose towards a contradiction that $A \notin \mathcal{H}^+$. Then there are $n \in \omega$ and $\alpha_1, \ldots, \alpha_n \in \omega_1$ for which $A \subseteq \bigcup_{i<n} H_n(\alpha_i)$. Without any loss of generality we may assume that there is $j < n$ so that, for any $i < n$, $\alpha_i \leqslant \beta$ if and only if $i < j$. Furthermore, we may assume that $n > \rho(\beta, \alpha_i)$ for any $i \geqslant j$. From this, it follows that

$$A \subseteq \Big(\bigcup_{i<j} H_n(\alpha_i) \Big) \cup \Big(\bigcup_{j \leqslant i < n} H_n(\alpha_i) \cap \beta \Big) \subseteq \Big(\bigcup_{i<j} H_n(\alpha_i) \Big) \cup \Big(\bigcup_{j \leqslant i < n} H_n(\beta) \Big).$$

In this way, $A \in \mathcal{H}|_\beta$ which is a contradiction.

$\square$

$\blacksquare$

Proposition 5.3.9. $\mathcal{H}$ *is Frechet in* $[\omega_1]^\omega$.

Proof. Let $A \in \mathcal{H}^+ \cap [\omega_1]^\omega$ and $\beta \in \omega_1$ be such that $A \subseteq \beta$. Note that $A \in (\mathcal{H}|_\beta)^+$. Since $\mathcal{H}|_\beta$ is a countably generated ideal, there is $B \in [A]^\omega$ such that $B \in (\mathcal{H}|_\beta)^\perp$. According to the point (a) of Lemma 5.3.8, $A \in \mathcal{I}$. Thus, the proof is over.

$\blacksquare$

Proposition 5.3.10. $\mathcal{I}$ *is a P-ideal.*

Proof. Let $\langle A_n \rangle_{n \in \omega}$ be an increasing sequence of elements of $\mathcal{I}$. Then there is $\beta \in \omega_1$ for which $\bigcup_{n \in \omega} A_n \subseteq \beta$. Let us enumerate $\beta + 1$ as $\langle \beta_n \rangle_{n \in \omega}$. We then define

$$A = \bigcup_{n \in \omega} \Big(A_n \backslash \Big(\bigcup_{i \leqslant n} H_n(\beta_i) \Big) \Big).$$

By the Lemma 5.3.5, it follows that A is a pseudo-union of $\langle A_n \rangle_{n \in \omega}$ and $A \in (\mathcal{H}|_\beta)^\perp$. In particular, this means that $A \in \mathcal{I}$ due to the point (a) of Lemma 5.3.8

$\blacksquare$

Theorem 5.3.11. $\mathcal{F}$ *is not* 2-*capturing.*

Proof. Suppose towards a contradiction that $\mathcal{F}$ is 2-capturing. According to the Proposition 5.3.10, $\mathcal{I}$ is a P-ideal. Since we are assuming PID, there are two possibilities for $\mathcal{I}$. We will finish the proof by showing that both of them lead to a contradiciton.

<u>Case 1</u>: There is an uncountable $Y \subseteq \omega_1$ such that $[Y]^\omega \subseteq \mathcal{I}$.

Non satisfaction of case. Consider the coloring $c : [Y]^2 \longrightarrow 2$ given by:

$$c(\{\alpha, \beta\}) = 0 \text{ if and only if } \{\alpha, \beta\} \text{ is captured.}$$

Using the partition relation $\omega_1 \longrightarrow (\omega_1, \omega + 1)^2_2$ and the fact that $\mathcal{F}$ is 2-capturing, we can conclude that there is an increasing sequence $\langle \alpha_i \rangle_{i < \omega + 1} \subseteq Y$ such that $\{\alpha_i, \alpha_j\}$ for any two distinct $i, j < \omega + 1$. In particular, $\{\alpha_i, \alpha_\omega\}$ is captured for any $i < \omega$. By lemma 2.3.9, we get that $A = \{\alpha_i : i < \omega\} \subseteq H_0(\alpha_\omega)$. In other words, $A \in \mathcal{I}$. This is a contradiction, so this case can not occur.

$\square$

Case 2: There Y an uncountable subset of ω_1 such that $[Y]^\omega \subseteq \mathcal{I}^\perp$.

Non satisfaction of case. According to Proposition 5.3.9, $\mathcal{I}$ is Frechet over $[\omega_1]^\omega$. From this it follows that $[Y]^\omega \subseteq \mathcal{H}$. Let

$$P = \{l \in \omega : \exists \alpha \in Y \, (\, \Xi_\alpha(l) > 0 \,)\}.$$

By refining Y, we may assume that for any $l \in P$ there are uncountable many $\alpha \in Y$ for which $\Xi_\alpha(l) > 0$. In this way, we can recursively construct sequences $\langle A_\beta \rangle_{\beta \in \omega_1} \subseteq [Y]^\omega$, $\langle B_\beta \rangle_{\beta \in \omega_1} \subseteq \mathrm{FIN}(\omega_1)$ and $\langle n_\beta \rangle_{\beta \in \omega_1}$ so that the following properties hold for any two $\beta < \gamma \in \omega_1$:

(1) $A_\beta < B_\beta < A_\gamma$. In particular $B_\beta \cap B_\gamma = \emptyset$.

(2) For all $l \in P$ there is $\alpha \in A_\beta$ for which $\Xi_\alpha(l) > 0$.

(3) $A_\beta \subseteq \bigcup_{\alpha \in B_\beta} H_{n_\beta}(\alpha)$.

(4) $B_\beta \cap Y \neq \emptyset$.

By refining such sequence, we can suppose that there is $n \in \omega$ such that $n_\beta = n$ for any $\beta \in \omega_1$. Since $\mathcal{F}$ is 2-capturing, there are $\beta < \gamma \in \omega_1$ such that $\{B_\beta, B_\gamma\}$ is captured at some level $l > n$. Now, B_β and B_γ are disjoint. Thus, $\Xi_\alpha(l) = 1$ for any $\alpha \in B_\gamma$. Therefore, $l \in P$ due to the point (4). By virtue of the point (2), there must be $\xi \in A_\beta$ such that $\Xi_\xi(l) > 0$. On the other hand, $\Xi_\alpha(l) = 0$ for all $\alpha \in B_\beta$. Consequently, $\xi \notin H_n(\alpha)$ for any such α. This contradicts the point (3), so this case can not occur.

$\square$

$\blacksquare$

Corollary 5.3.12. *(Under PID) There are no 2-capturing construction schemes.*

Chapter 6

Open problems

6.1 Problems about gaps

One of the most important concepts introduced in Section 3.1 was the Luzin representation of a partial order and what it means for a Luzin family to code a partial order (see Definition 3.1.15) Indeed, in order to prove that there is a Luzin-Jones family $\mathcal{A}$ in ZFC, we ended up proving that such family $\mathcal{A}$we can differentiatet is natural to ask if the same is true for any Luzin-Jones family.

Problem 6.1.1. Is there a Luzin-Jones family which doesn't code ω_1?

A partial answer to this question is "no". If $\mathfrak{b} > \omega_1$ then any Luzin family is Jones and it codes ω_1. Thus, a related problem would be whether the existence of a Luzin-Jones family which doesn't code ω_1 follows from the equality $\mathfrak{b} = \omega_1$. Although such a family might exist, it is still possible that there is a partial order X which characterizes the Jones property in terms of coding: Formally:

Problem 6.1.2. Is there a partial order X so that for any Luzin family, coding X and being Jones are equivalent statements?

Given an almost disjoint family $\mathcal{A}$, let us define the *spectrum of* $\mathcal{A}$, namely $Spec(\mathcal{A})$, as the class of all partial orders $(X, <)$ such that $\mathcal{A}$ codes X (Definition 3.1.15). The class $Spec(\mathcal{A})$ can be regarded as a measurement of the "malleability" of $\mathcal{A}$. From the point of view of the author, it is an interesting problem to know how much we can differentiate an almost disjoint family from another one by analysing their spectrums.

Problem 6.1.3. Which are the possible values of $Spec(\mathcal{A})$ for a given Luzin family $\mathcal{A}$?

Let $\mathcal{X}_{\omega_1}$ denote the class of ω_1-like orders. In Theorem 3.1.20, we have seen that the Luzin-Jones family $\mathcal{A}$ that we constructed codes each element of $\mathcal{X}_{\omega_1}$. Following with Problem 6.1.1, we may ask how strong is the property of coding each ω_1-like order in relation to the property of being Luzin-Jones. We may ask also ask how much of the spectrum do we need to know in order to assure that it contains $\mathcal{X}_{\omega_1}$. That is:

Problem 6.1.4. Is there a finite family $\mathcal{Y} \subseteq \mathcal{X}_{\omega_1}$ so that, for any Luzin-Jones family $\mathcal{A}$, if $\mathcal{Y} \subseteq Spec(\mathcal{A})$, then $\mathcal{X}_{\omega_1} \subseteq Spec(\mathcal{A})$?

Now we move to problems about gaps in a more general context. It is a theorem of Rothberger that $\mathfrak{b}$ is the minimal cardinal κ for which (ω, κ)-gaps exist. In fact, when we restrict to cardinals, the only gaps that we can construct in ZFC alone are the ones of type (ω_1, ω_1), $(\omega, \mathfrak{b})$ and $(\mathfrak{b}, \omega)$. Indeed, if we assume PFA then these are the only posibilities (see [113]). However, as we have shown in Theorem 3.1.21, the situation becomes much more complex when we consider (X, Y)-gaps where X and Y have cofinality ω_1. We wonder if the same level of complexity carries to (X, Y)-gaps when $|X| = \omega$ and the cofinality of Y is $\mathfrak{b}$. Before stating our problem, let us consider the natural generalisation of orders similar to ω_1.

Definition 6.1.5. Let κ be a cardinal and $(X, <)$ be a partial order. Let us call $(X, <)$ a κ-*like order* whenever X is well-founded, $|X| = \kappa$, and $|(-\infty, x)| < \kappa$ for each $x \in X$.

We then ask:

Problem 6.1.6. Is there is an (ω, X)-gap for any $\mathfrak{b}$-like order X? Does PFA decides this question?

It can been seen that the answer to this question is "yes" under CH.

The study of the Gap Cohomology group of $*$-lower semi-lattices was originally motivated due to the work of Talayco regarding towers. We have already mentioned that Todorčević showed that $|\mathcal{G}(\mathcal{T})| = 2^{\omega_1}$ for any ω_1-tower $\mathcal{T}$[1]. Thus, it is natural to ask whether the same result holds for any $*$-lower semi-lattice. A partial answer to this question lies in almost disjoint families known as strong-Q-sequences (also known as uniformizable AD-systems). For more about these objects see [19], [94] and [104].

Definition 6.1.7 (Strong-Q-sequence). Let $\mathcal{A}$ be an almost disjoint family. We say that $\mathcal{A}$ is a *strong-Q-sequence* if for each sequence $\langle f_A \rangle_{A \in \mathcal{A}}$ with $f : A \longrightarrow 2$ for each $A \in \mathcal{A}$, there is $f : \omega \longrightarrow 2$ such that:

$$f|_A =^* f_A$$

for each $A \in \mathcal{A}$.

It is easy so see that the existence of a strong-Q-sequence of size ω_1 (in $\mathscr{P}(\omega)/\mathrm{FIN}$) is equivalent to the existence of a $*$-Lower semi-lattice $\mathcal{T}$ isomorphic to $[\omega_1]^{<\omega}$ for which the gap cohomology group of $\mathcal{T}$, namely $\mathcal{G}(\mathcal{T})$, is trivial. Thus, Todorčević result about the size of the gap cohomology group of towers cannot be extended further without assuming additional axioms. The following problems arise:

Problem 6.1.8. Is there (in ZFC) an ω_1-like $*$-lower semi-lattice $\mathcal{T}$ such that $|\mathcal{G}(\mathcal{T})| = \omega$?

---

[1]Recall that our towers do not need to be maximal.

Problem 6.1.9. Is it consistent that $\mathcal{G}(\mathcal{T})$ is non-trivial for any ω_1-like $*$-lower semi-lattice $\mathcal{T}$?

Problem 6.1.10. Let X be an ω_1-like lower semi-lattice and suppose that $\mathcal{G}(\mathcal{T})$ is non-trivial for any $*$-lower semi-lattice $\mathcal{T}$ isomorphic to X. Is it true that $|\mathcal{G}(\mathcal{T})| = 2^{\omega_1}$ for any such $\mathcal{T}$?

As a direct corollary of Theorem 3.1.88, we have that FCA implies the existence of a family of size $\mathfrak{c}$ of independent gaps. Note that the main result of Yorioka in [133] under $\diamondsuit$ is stronger than ours. Thus, we ask:

Problem 6.1.11. Does FCA imply the existence of a family of size 2^{ω_1} of independent gaps? What about $FCA(part)$?

6.2 Problems about trees and lines

An uncountable linear order $(X, <)$ is said to be *minimal* if for each uncountable $Y \subseteq X$, there is a strictly increasing function from X to Y. It is a well-known theorem that $\mathfrak{m} > \omega_1$ implies that every Countryman line is minimal (see [93], [77], [10] and [109]). Is this statement true when we use any of the parameterized Martin's axioms $\mathfrak{m}_{\mathcal{F}}^n$ instead of $\mathfrak{m}$?

Problem 6.2.1. Let $2 \leqslant n \in \omega$ and $\mathcal{F}$ be an n-capturing construction scheme. Suppose that $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$. Is it true that any Countryman line is minimal?

In this thesis, we showed that given a construction scheme $\mathcal{F}$, there is a natural way to define an order $<_{\mathcal{F}}$ over ω_1 such that $(\omega_1, <_{\mathcal{F}})$ is a Countryman line (see Definition 3.2.7). So even if Problem 6.2.1 has a negative answer, it is interesting to know whether $(\omega_1, <_{\mathcal{F}})$ is minimal or not.

Problem 6.2.2. Let $2 \leqslant n \in \omega$ and $\mathcal{F}$ be an n-capturing construction scheme. Suppose that $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$. Is it true that $(\omega_1, <_{\mathcal{F}})$ is a minimal Countryman line?

One of the main applications of the cardinal $\mathfrak{m}_{\mathcal{F}}$ was to show that under $\mathfrak{m}_{\mathcal{F}} > \omega_1$, there are no Suslin trees. Baumgartner proved something stronger by using the cardinal $\mathfrak{m}$. He proved that under $\mathfrak{m} > \omega_1$, every Aronszajn tree is special. Thus, a natural question is the following:

Problem 6.2.3. Let $\mathcal{F}$ be a 2-capturing construction scheme. Does the inequality $\mathfrak{m}_{\mathcal{F}} > \omega_1$ implies that every Aronszajn tree is special?

In some sense, the non-existence of Suslin trees under $\mathfrak{m}_{\mathcal{F}} > \omega_1$ says that the property of being 2-capturing is not strong enough to imply the existence of a Suslin tree. On the other hand FCA does imply the existence of a Suslin tree. So how much capturing is it really needed for this result?

Problem 6.2.4. Let $3 \leqslant n \in \omega$ and $\mathcal{F}$ be a 3-capturing construction scheme. Does the inequality $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$ imply the Suslin's hypothesis?

The same question applies to entangled sets.

Problem 6.2.5. Does the existence of 2-entangled sets follow from either CA_n or CA? In other words, which is the weakest capturing axiom implying the existence of 2-entangled sets?

6.3   More problems regarding applications

In Theorem 3.3.9 we proved that, from CA_3, we can construct a coloring $c : [\omega_1]^2 \longrightarrow 2$ which is 2-bounded, has no uncountable injective sets, but is *ccc* destructible with respect to this last property.

Problem 6.3.1. Does the existence of a coloring such as the one in Theorem 3.3.9 follow from CA_2? Given a construction scheme $\mathcal{F}$, does $\mathfrak{m}_{\mathcal{F}} > \omega_1$ imply the non-existence of such a coloring?

Regarding Tukey types, we propose the following question.

Problem 6.3.2. How many distinct Tukey types on ω_1 exist under CA_2?

Particularly, we can ask:

Problem 6.3.3. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are distinct 2-capturing construction schemes. Find sufficient conditions under which $(\mathcal{B}_{\mathcal{F}}, \leqslant)$ is incomparable with $(\mathcal{B}_{\mathcal{F}'}, \leqslant)$.

Under Martin's Axiom, a great variety of ultrafilters can be constructed. Is the same true for $\mathfrak{m}_{\mathcal{F}}$?

Problem 6.3.4. Let $2 \leqslant n \in \omega$ and $\mathcal{F}$ be an n-capturing construction scheme. Does $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$ imply the existence of union ultrafilters?

Let $S \subseteq \omega_1$ be stationary. Recall that a *ladder system* on S is a sequence $\langle L_\alpha \rangle_{\alpha \in S}$ such that L_α is an unbounded subset of α of order type ω for each $\alpha \in S$.

Definition 6.3.5 (Uniformization of ladders). A ladder system $\langle L_\alpha \rangle_{\alpha \in S}$ is said to be *uniformizable* if of each sequence $\langle f_\alpha \rangle_{\alpha \in S}$ of functions $f_\alpha : L_\alpha \longrightarrow \omega$, there is $F : \omega_1 \longrightarrow \omega$ such that:

$$f_\alpha =^* f$$

for any $\alpha \in S$.

It is known that $\mathfrak{m} > \omega_1$ implies that every ladder system is uniformizable. On the other hand, $2^\omega < 2^{\omega_1}$ implies that no ladder system is uniformizable (see [27]). Throughtout the course of the investigation of construction schemes, we tried to construct ladder systems with interesting properties. In particular, we tried to construct a ladder system which is no uniformizable by using FCA. Nevertheless we didn't succeed in that task. So:

Problem 6.3.6. Does the existence of a non-uniformizable ladder system follow from FCA? What about $FCA(part)$?

Problem 6.3.7. Let $2 \leqslant n \in \omega$ and $\mathcal{F}$ be an n-capturing construction scheme. Does $\mathfrak{m}_{\mathcal{F}}^n > \omega_1$ imply that every ladder system is uniformizable?

To learn more about uniformizability of ladder systems see [8] and [23].

6.4 Problems about construction schemes

Recently, in [88], Assaf Rinot and Roy Shalev introduced the principle $\clubsuit_{AD}$. This principle follows both from $\clubsuit$ and from the existence of a Suslin Tree. Thus, it also follows from FCA. In this way, we may ask:

Problem 6.4.1. Is any instance of $\clubsuit_{AD}$ a consequence of CA_n for some $2 \leqslant n \in \omega$? What about $CA_n(part)$?

One of the most famous problems outside set theory, in which set theory played an important role, was the solution of Naimark's problem regarding C^*-algebras. In [5], Charles Akemann and Nik Weaver constructed a counterexample to Naimark's problem using $\Diamond$-principle. In [18], Daniel Calderón and Ilijas Farah isolated the guessing principle $\Diamond^{Cohen}$, which follows from $\Diamond$. There, they constructed a counterexample to Naimark's problem using $\Diamond^{Cohen}$.

Problem 6.4.2. Is CA_n, CA or FCA a consequence of $\Diamond^{Cohen}$? What about $CA_n(part)$, $CA(part)$ and $FCA(part)$?

Problem 6.4.3. Does the existence of a counterexample to Naimark's problem follow from $FCA(part)$?

In [78], Mirna Džamonja, Michael Hrušák and Justin Moore introduced parametrized diamonds, which are weakenings of $\Diamond$ that may hold in models where CH fails. Both capturing construction schemes and parametrized diamonds are useful tool for building interesting objects of size ω_1. We ask:

Problem 6.4.4. Is there any relation between FCA (respectively CA_n) and any parametrized $\Diamond$-principle?

It would be specially interesting to know if there is any realtionship between FCA and $\Diamond(non(\mathcal{M}))$.

We already know that the forcing $\mathbb{C}_\kappa$ forces $FCA(part)$ whenever κ is an uncountable ordinal. But what happens when we only add one Cohen real?

Problem 6.4.5. Does the Cohen forcing $\mathbb{C}$ force FCA? What about $FCA(part)$?

By far, the most interesting problem from the point of view of the author is the following:

Problem 6.4.6. Extend the theory of capturing construction schemes to higher cardinals.

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